Massachusetts Institute of Technology
Department of Mechanical Engineering

2.001 Spring 2011 **Quiz 1** **October 13th, 2011**

Name: ____________________

Recitation: ____________________

Answer all questions in the booklet provided. **Be sure your name and recitation section are marked BOTH on the cover of your booklet and on this exam.** Partial credit will be awarded so be sure and show your work - this includes drawing *clear* and *well-labeled* diagrams. You may assume small deformations and linear elastic materials in all problems. You may bring two double-sided sheets of notes; NO CALCULATORS.

This quiz will test the following skills/concepts in the context of axially loaded structures:

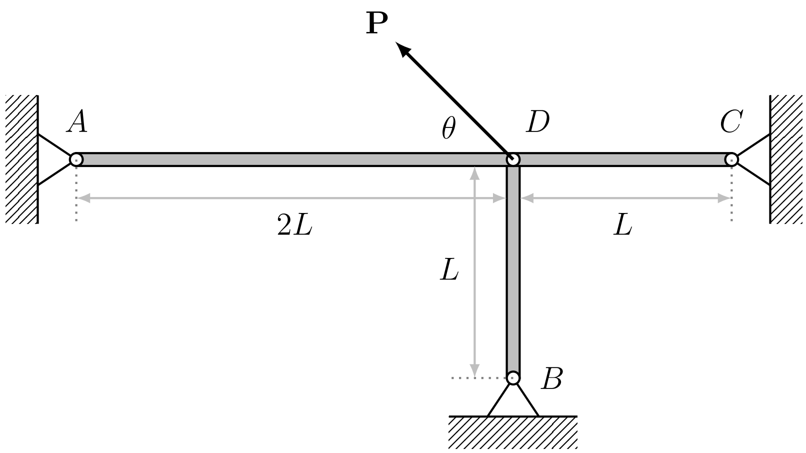
- Applying the concepts of equilibrium
- Applying concepts of compatibility (particularly small deformation approximations)
- Demonstrate an understanding of constitutive relationships
- Ability to model composite structures
- Distinguishing between statically determinate and statically indeterminate structures
- Identifying degrees of freedom
- Demonstrate an understanding of the concepts of stress and strain
- Ability to perform algebra and integrate
- Obtaining answers with appropriate units, signs and trends

Each question will require you to use a subset of these skills. For each question, we will grade each of the relevant concepts/skills on a scale of 0-3.

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.001 Spring 2011 **Quiz 1** **October 13th, 2011**

Problem 1



Consider the above system of three bars connected by ideal hinges to each other at point D and to fixed points A, B, and C on the ground. Each bar has the same Young's modulus E and cross sectional area $\mathcal{A}$, with lengths specified above. A force $\mathbf{P}$ is applied at point D at an angle θ from the horizontal as shown.

(a) Is this structure statically determinate or statically indeterminate? Please explain.

(b) Determine the displacement of point D assuming small displacements.

(c) Which beam ($\overline{AD}$, $\overline{BD}$, or $\overline{CD}$) experiences the highest magnitude stress assuming $\theta = 45°$?

Problem 2

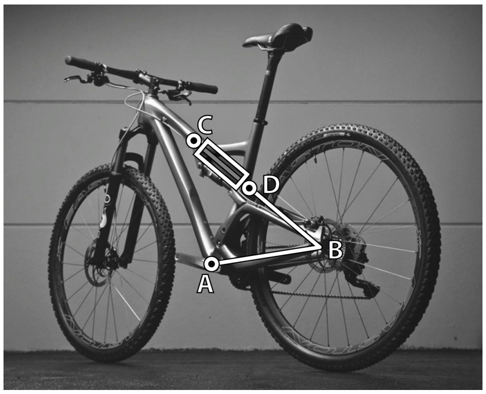


Figure (a)

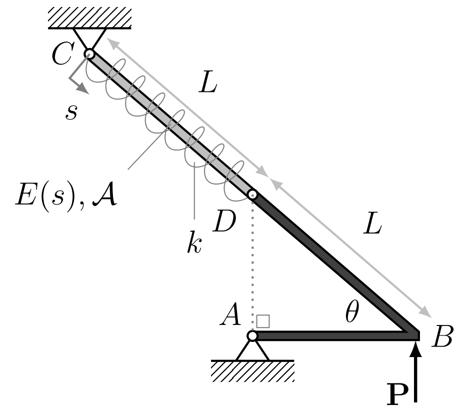


Figure (b)

A popular design for mountain bike suspensions is shown in Figure (a). The rear suspension can be modeled as a deformable shock absorber CD connected to a rigid "V", idealized in Figure (b) as rigid body ABD. A heavy-duty shock absorber often consists of two components: an air-shock plus a steel coil. For this analysis, we will model the coil as an ideal spring with spring constant k and un-stretched length L, and the air-shock as a deformable core with constant cross-sectional area $\mathcal{A}$ and variable Young's modulus $E(s) = E_0(1 + s/L)$ where s varies along the length of the core for $0 \leq s \leq L$. Now imagine that the rear wheel of our bike experiences some vertical force $\mathbf{P}$ while you ride (while hitting rocks and such).

(a) Consider the shock absorber assembly. Is it a two-force member?

(b) How many degrees of freedom are there in this structure? What are they?

(c) Assuming the structure is in equilibrium, find the axial load Q on the shock absorber assembly in terms of given quantities.

(d) Write an equation relating Q to the internal axial forces $\mathcal{N}_S$ and $\mathcal{N}_C$ acting on the spring and air-shock core respectively.

(e) Determine the elongation δ of the shock absorber assembly in terms of given quantities (i.e. NOT in terms of Q, $\mathcal{N}_S$, $\mathcal{N}_C$).

(f) Derive an expression relating your degree(s) of freedom from part (b) to the elongation δ of the shock absorber assembly.

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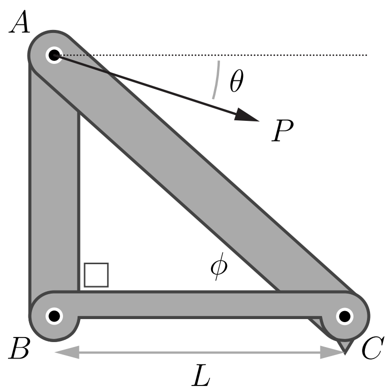
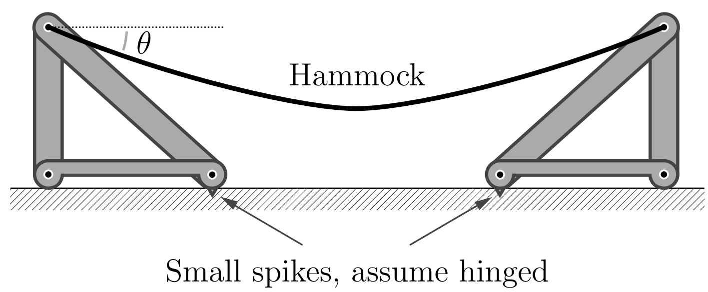
Problem 3



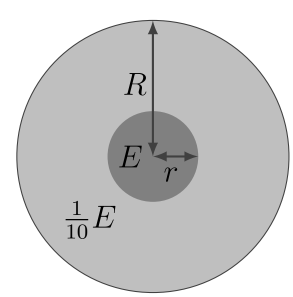


The following design has been proposed for a low-cost, modular hammock. It consists of two separate truss components connected by a hammock. Each truss has length L and height $L \tan\phi$. Small spikes on each truss emulate ideal hinges when pressed into the ground. Consider the mass of the truss components to be negligible compared to the force exerted on the truss by the hammock, and consider any frictional force occurring at point B to be negligible compared to any normal reactions.

(a) Let P be the force the hammock exerts on each truss. Draw a free body diagram of the *left* truss with all relevant reaction forces. Do NOT include forces that have been assumed to be negligible. Solve for what these reaction forces must be as functions of θ.

(b) What conditions on angle θ must exist in order for the truss not to tip? Hint: What can you say about the reaction forces if the truss begins to pivot about point C?

(c) The figure to the right shows a cross section of the cable that attaches the hammock to point A while loaded with force P. Two materials are used for the wire and coating to enable a strong cable at low cost. A design goal is to make each material carry the same load. Given that the radius r of the wire is fixed and the Young's moduli of the wire and coating are E and $\frac{1}{10}E$ respectively, what radius R must the coating be in order to have both materials carry the same load?



Solutions: Problem 1

(a) Is this structure statically determinate or statically indeterminate? Please explain.

This system is statically indeterminate because the structure would remain in equilibrium if beam $\overline{AD}$ were removed.

(b) Determine the displacement of point D assuming small displacements.

Let $\overline{AD}$ be bar 1, $\overline{BD}$ be bar 2, $\overline{CD}$ be bar 3, and N_1, N_2, and N_3 be the internal axial loadings in each beam respectively. This system has two degrees of freedom, the x and y positions u_x^D, u_y^D of point D. Consider equilibrium on the system isolating hinge D:

$$\sum F_x^D = 0 = N_3 - N_1 - P\cos\theta$$
$$\sum F_y^D = 0 = P\sin\theta - N_2$$

Constitutive relations for uniform two force members relates elongations to internal axial loads:

$$\delta_1 = \frac{N_1(2L)}{EA}, \quad \delta_2 = \frac{N_2 L}{EA}, \quad \delta_3 = \frac{N_3 L}{EA}$$

Compatibility yields relations between the degrees of freedom with the elongations. Since each degree of freedom is either perpendicular or parallel to each bar, the relationships are straight forward:

$$\delta_1 = u_x^D, \quad \delta_2 = u_y^D, \quad \delta_3 = -u_x^D$$

We have 8 equations in 8 unknowns (u_x^D, u_y^D, N_1, N_2, N_3, δ_1, δ_2, δ_3). Solving for (u_x^D, u_y^D) gives:

$$\boxed{(u_x^D, u_y^D) = \frac{PL}{EA}\left(-\frac{2}{3}\cos\theta, \sin\theta\right)}$$

(c) Which beam ($\overline{AD}$, $\overline{BD}$, or $\overline{CD}$) experiences the highest magnitude stress assuming $\theta = 45°$?

Beam $\overline{BD}$ experiences the highest magnitude stress as it has the longest elongation and the shortest length, all else being equal.

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Solutions: Problem 2

(a) Consider the shock absorber assembly. Is it a two-force member?

$\boxed{\text{Yes}}$, the shock absorber assembly is a two-force member.

(b) How many degrees of freedom are there in this structure? What are they?

This structure has $\boxed{\text{one}}$ degree of freedom and can be defined by the positive rotation $\boxed{\text{angle } \phi}$ of the rigid "V" about point A.

(c) Assuming the structure is in equilibrium, find the axial load Q on the shock absorber assembly in terms of given quantities.

Consider the isolated system of the rigid "V". Take moment equilibrium about point A:

$$\sum (M_z)_A = 0 = PL\cos\theta + QL\cos\theta\sin\theta \quad \Longrightarrow \quad \boxed{Q = -\frac{P}{\sin\theta}}$$

(d) Write an equation relating Q to the internal axial forces $\mathcal{N}_S$ and $\mathcal{N}_C$ acting on the spring and air-shock core respectively.

$$\boxed{Q = \mathcal{N}_S + \mathcal{N}_C}$$

(e) Determine the elongation δ of the shock absorber assembly in terms of given quantities (i.e. NOT in terms of Q, $\mathcal{N}_S$, $\mathcal{N}_C$).

Constitutive relations relate elongations to internal axial forces. For the spring, $N_S = k\delta$, thus $N_C = -P/(\sin\theta) - k\delta$. For the air-shock core, we have:

$$\delta = \int_L \frac{N_C}{\int_A E\,dA}\,dL = \int_0^L \frac{N_C}{E_0(1+s/L)A}\,ds = \frac{N_C L}{E_0 A}\left[\ln\left|1+\frac{s}{L}\right|\right]_0^L = \frac{N_C L}{E_0 A}\ln|2|$$

$$\delta = -(P/(\sin\theta) + k\delta)\frac{L}{E_0 A}\ln|2| \quad \Longrightarrow \quad \boxed{\delta = -\frac{PL\ln|2|}{(E_0 A + kL\ln|2|)\sin\theta}}$$

(f) Derive an expression relating your degree(s) of freedom from part (b) to the elongation δ of the shock absorber assembly.

From this configuration, positive small angle movement of ϕ yields a displacement of point D to the left by a distance $\phi L\sin\theta$. Compatibility yields:

$$\boxed{\delta = -\phi L\sin\theta\cos\theta}$$

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DEPARTMENT OF MECHANICAL ENGINEERING

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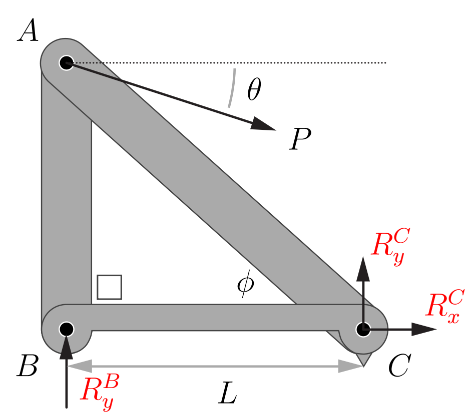
Solutions: Problem 3

(a) Let P be the force the hammock exerts on each truss. Draw a free body diagram of the *left* truss with all relevant reaction forces. Do NOT include forces that have been assumed to be negligible. Solve for what these reaction forces must be as functions of θ.

This is a determinate system, thus equilibrium on the isolated truss structure yields three equations in all three unknowns:



$$\sum F_x = 0 = R_x^C + P\cos\theta$$

$$\sum F_y = 0 = R_y^C + R_y^B - P\sin\theta$$

$$\sum (M_z)_C = 0 = -\left[\frac{L}{\cos\phi}\sin(\phi-\theta)\right]P - (L)R_y^B$$

Yields:

$$\boxed{R_x^C = -P\cos\theta} \quad \boxed{R_y^B = -P\frac{\sin(\phi-\theta)}{\cos\phi}} \quad \boxed{R_y^C = P\sin\theta + P\frac{\sin(\phi-\theta)}{\cos\phi} = P\cos\theta\tan\phi}$$

(b) What conditions on angle θ must exist in order for the truss not to tip? Hint: What can you say about the reaction forces if the truss begins to pivot about point C?
For the truss not to tip, $R_y^B \leq 0$, thus $\sin(\phi-\theta) \leq 0$, and $\boxed{\phi \leq \theta \leq \pi/2}$

(c) The figure to the right shows a cross section of the cable that attaches the hammock to point A while loaded with force P. Two materials are used for the wire and coating to enable a strong cable at low cost. A design goal is to make each material carry the same load. Given that the radius r of the wire is fixed and the Young's moduli of the wire and coating are E and $\frac{1}{10}E$ respectively, what radius R must the coating be in order to have both materials carry the same load?

$$\mathcal{N}_W = \mathcal{N}_C \qquad \delta_W = \delta_C \qquad \frac{\delta_W}{L} = \frac{\mathcal{N}_W}{E\pi r^2} \qquad \frac{\delta_C}{L} = \frac{10\,\mathcal{N}_C}{E\pi(R^2-r^2)}$$

Thus, $10\,r^2 = R^2 - r^2$, yielding: $\boxed{R = r\sqrt{11}}$

Massachusetts Institute of Technology

Department of Mechanical Engineering

2.001 Spring 2011 Quiz 2 November 10th, 2011

Answer all questions in the booklet provided. **Be sure your name and recitation section are marked on the cover of your booklet. This question book will NOT be graded.** Partial credit will be awarded so be sure and show your work - this includes drawing *clear* and *well-labeled* diagrams. You may assume small deformations and linear elastic materials in all problems. You may bring two double-sided sheets of notes; NO CALCULATORS.

This quiz will test the following skills/concepts:

- Applying the concepts of equilibrium
- Applying concepts of compatibility
- Demonstrate an understanding of constitutive relationships
- Ability to model composite structures
- Ability to model thermal expansion
- Demonstrate an understanding of the concepts of stress and strain
- Ability to perform algebra and integrate
- Obtaining answers with appropriate units, signs and trends

Each question will require you to use a subset of these skills. For each question, we will grade each of the relevant concepts/skills on a scale of 0-3.

Massachusetts Institute of Technology
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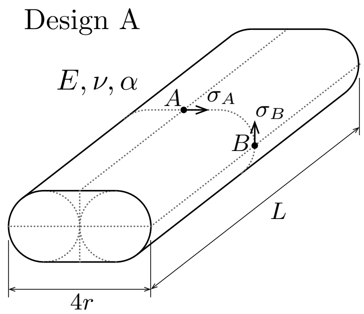
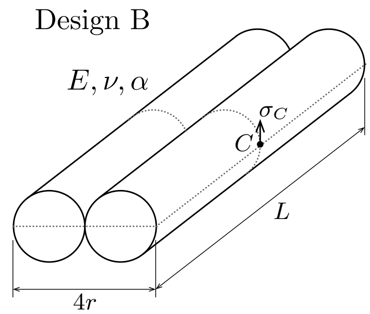
2.001 Spring 2011 **Quiz 2** **November 10th, 2011**

Problem 1



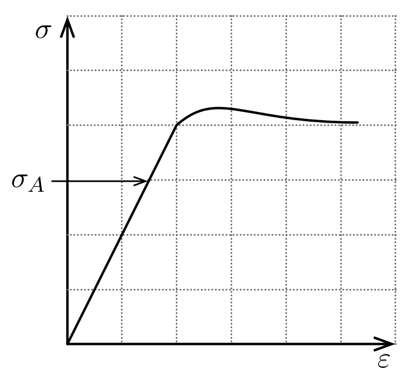


Two tank designs have been proposed to transport an explosive, corrosive, radioactive compressed gas via semi-truck. Design B is made from two separate cylindrical tanks side by side. The engineers are worried that the stress in design A will be too high, hence design B. The gas is maintained at constant inner temperature T_i via an internal system and can be modeled as an ideal gas ($PV = nRT =$ constant). Each design must carry the same amount of gas, so since design A has greater volume, design B must hold the gas at higher pressure ($P_A V_A = P_B V_B$). The skin of the tank is at the average of the outside temperature T_o and T_i. Assume the vessel can be accurately modeled as a thin-walled pressure vessel of thickness t. When unpressurized, the vessels experience zero strain at T_i.

(a) **CQ:** In design A, is the hoop stress larger at point A or point B? DO NOT use equations from part (b) to answer part (a). We are looking for a one or two sentence physical explanation that argues conceptually which stress is larger based on geometry.

(b) Use free body diagrams to find the hoop stresses at points A, B, and C in terms of P_A, E, ν, r, L and t. Make sure to use clearly labeled free body diagrams of the cross sections you consider so that we understand your process.

(c) It is important to prevent yield of the tank wall. The stress-strain relationship for the tank material is shown at right. Will either tank design yield?



(d) Consider one of the tanks in design B. If the ends of the tank are mounted in a way that prevents only axial elongation, find T_o at which yield would occur in the tank.

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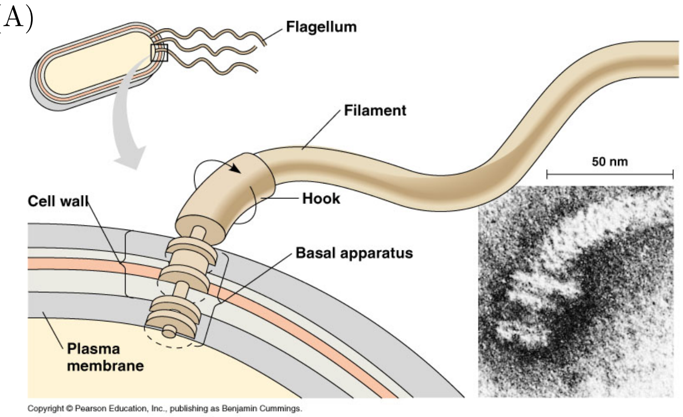
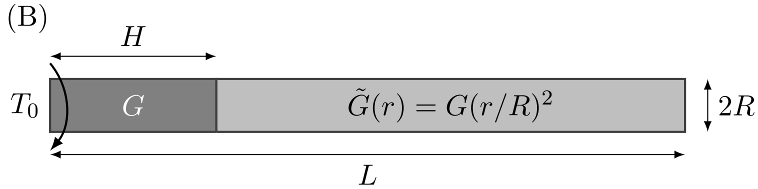
Problem 2

(A)



(B)



Bacteria such as *E. coli* swim by rotating a long slender tail that is connected to the cell body via a rotary motor which applies a constant torque T_0 in the direction indicated in Figure B. The tail is composed of a "hook" attached to a long slender structure. In this problem, we will consider a simplified tail model of constant radius R, composed of a hook with length H and constant shear modulus G, and a slender structure with length $L-H$ and variable shear modulus $\tilde{G}(r) = G(r/R)^2$.

(a) The torque from the motor is resisted by viscous drag from the fluid which supplies a constant distributed torque per unit length q along the tail. If the tail is in equilibrium, derive an expression for q in terms of known quantities.

(b) Compute and sketch the internal torque $T(x)$ as a function of x. Draw a clear, well-labeled free body diagram.

(c) Compute and sketch the twist angle $\phi(x)$ as a function of x. You may assume $\phi = 0$ at the motor.

(d) Sketch the maximum stress in the tail as a function of x. Under what conditions does the maximum magnitude of stress occur in the hook? We are looking for a geometric relationship between L and H.

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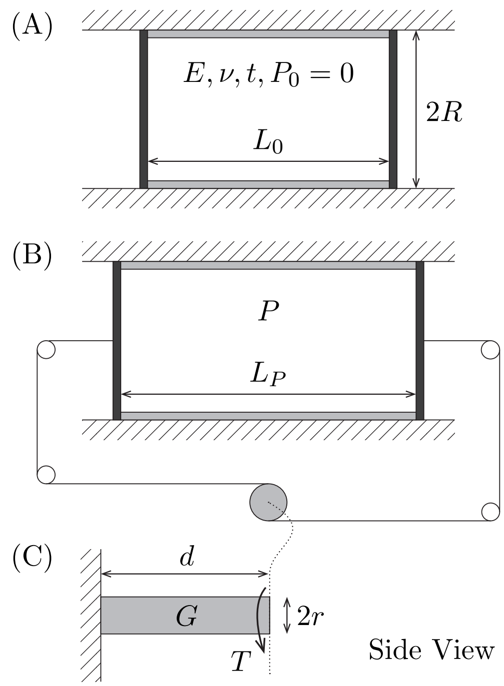
Problem 3

A thin-walled pressure vessel, made from the combination of a deformable tube (material properties E and ν) and two rigid end caps, has unpressurized radius R, thickness t, and length L_0. It is inserted into a frictionless rigid pipe with the same internal radius. Throughout the problem, assume temperatures remain constant. (Figure A)



(a) Gas is pumped into the vessel to a gauge pressure $P > 0$. What is the pressurized length L_P of the cylinder in terms of known quantities?

After the vessel has been pressurized inside the pipe (Figure B), each end is attached via inextensible cables around pulleys to a deformable rod of radius r, length d, and uniform shear modulus G, which itself is attached to a wall. (Figure C)

The vessel is then completely depressurized, causing the vessel to shrink to a length L_F with $L_0 < L_F < L_P$, and causing the deformable rod to rotate by an angle ϕ. Solve this statically indeterminate system by answering the following questions:

(b) What is the tension F in the cables in terms of the final length L_F and known quantities?

(c) What is the torque T acting on the deformable bar in terms of the tension F in the cables and known quantities?

(d) What is the twist angle ϕ of the deformable bar in terms of the applied torque T and known quantities?

(e) Write a compatibility equation relating the final length L_F, the pressurized length L_P, the deflection angle and known quantities.

(f) Justify that these equations are enough to solve for any of the unknowns.

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Solutions: Problem 1

(a) **CQ:** In design A, is the hoop stress larger at point A or point B? DO NOT use equations from part (b) to answer part (a). We are looking for a one or two sentence physical explanation that argues conceptually which stress is larger based on geometry.

From the geometry of the tank, the tank material available to resist the pressure in the vertical direction (resisted by σ_B) and horizontal direction (resisted by σ_A) is the same. However the pressure in the vertical direction acts over a larger area than in the horizontal direction so the stress in the vertical direction must be larger.

(b) Use free body diagrams to find the hoop stresses at points A, B, and C in terms of P_A, E, ν, r, L and t. Make sure to use clearly labeled free body diagrams of the cross sections you consider so that we understand your process.

$$P_A V_A = P_B V_B \implies P_A(\pi r^2 + 4r^2)L = P_B(2\pi r^2)L \implies P_B = P_A\left(\frac{\pi+4}{2\pi}\right)$$

$$\boxed{\sigma_A = \frac{P_A r}{t}} \qquad \boxed{\sigma_B = \frac{2P_A r}{t}} \qquad \boxed{\sigma_C = \frac{P_A r}{t}\left(\frac{\pi+4}{2\pi}\right)}$$

(c) It is important to prevent yield of the tank wall. The stress-strain relationship for the tank material is shown at right. Will either tank design yield?

$$\sigma_B = 2\sigma_A > \frac{4}{3}\sigma_A \qquad \sigma_C = \left(\frac{\pi+4}{2\pi}\right)\sigma_A \approx 1.14\sigma_A < \frac{4}{3}\sigma_A$$

Thus, design A will yield while design B will not.

(d) Consider one of the tanks in design B. If the ends of the tank are mounted in a way that prevents only axial elongation, find T_o at which yield would occur in the tank.

Yield will occur when the axial stress $\sigma_{xx} = (4/3)\sigma_A$. Thus, since:

$$\varepsilon_{xx} = 0 = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{\theta\theta} + \cancelto{0}{\sigma_{rr}})] + \alpha\Delta T \implies 0 = \frac{4}{3}\sigma_A - \nu\left(\frac{\pi+4}{2\pi}\right)\sigma_A + \alpha E\left(\frac{T_o - T_i}{2}\right)$$

$$\boxed{T_o = T_i + \frac{2P_A r}{Et\alpha}\left[\nu\left(\frac{\pi+4}{2\pi}\right) - \frac{4}{3}\right]}$$

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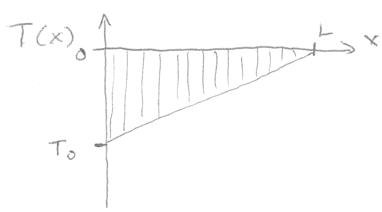
Solutions: Problem 2

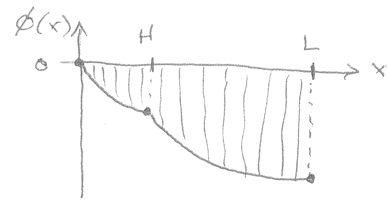
(a) The torque from the motor is resisted by viscous drag from the fluid which supplies a constant distributed torque per unit length q along the tail. If the tail is in equilibrium, derive an expression for q in terms of known quantities.

$$\boxed{q = -\frac{T_0}{L}}$$

(b) Compute and sketch the internal torque $T(x)$ as a function of x. Draw a clear, well-labeled free body diagram.

$$\sum M_x = 0 = T(x) + T_0 + qx \quad \Longrightarrow \quad \boxed{T(x) = T_0(x/L - 1)}$$





(c) Compute and sketch the twist angle $\phi(x)$ as a function of x. You may assume $\phi = 0$ at the motor.

$$\phi(x) = \int_0^x \frac{T(x')}{(GI_P)_{eff}} dx'$$

For $0 \leq x \leq H$:

$$(GI_P)_{eff} = \frac{\pi}{2} G R^4 \quad \Longrightarrow \quad \boxed{\phi(0 \leq x \leq H) = \frac{2T_0}{\pi G R^4}\left(\frac{x^2}{2L} - x\right)}$$

For $H \leq x \leq L$:

$$(GI_P)_{eff} = \int_0^{2\pi}\int_0^R \left[G\left(\frac{r}{R}\right)^2 r^2\right] r\,dr\,d\theta = \frac{\pi}{3} G R^4 \quad \Longrightarrow \quad \phi(H \leq x \leq L) = \frac{3T_0}{\pi G R^4}\left(\frac{x^2}{2L} - x\right) + \phi(H)$$

$$\boxed{\phi(H \leq x \leq L) = \frac{3T_0}{\pi G R^4}\left[\frac{x^2 - H^2}{2L} - (x - H)\right] + \frac{2T_0}{\pi G R^4}\left(\frac{H^2}{2L} - H\right)}$$

(d) Sketch the maximum stress in the tail as a function of x. Under what conditions does the maximum magnitude of stress occur in the hook? We are looking for a geometric relationship between L and L_1.

$$\varepsilon_{x\theta} = \frac{\sigma_{x\theta}}{2G(x,r)} = \frac{r}{2}\frac{d\phi}{dx} = \frac{r}{2}\frac{T(x)}{(GI_P)_{eff}} \implies \sigma_{x\theta} = \frac{r\tilde{G}(x,r)T(x)}{(GI_P)_{eff}}$$

In hook:

$$|\sigma_{x\theta}^{hook}| = \left|\frac{2rT_0}{\pi R^4}(x/L-1)\right|$$

Maximum when $r = R$ and $x = 0$, so max stress occurs at $|\sigma_{x\theta}^{hook}|_{max} = 2T_0/(\pi R^3)$

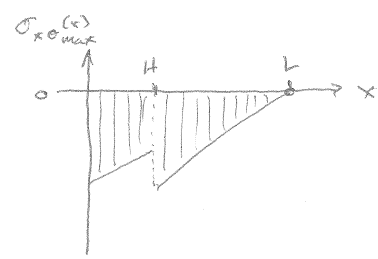
In tail:

$$|\sigma_{x\theta}^{tail}| = \left|\frac{3rT_0}{\pi R^4}(x/L-1)\right|$$

Maximum when $r = R$ and $x = H$, so max stress occurs at $|\sigma_{x\theta}^{hook}|_{max} = 3T_0(1-H/L)/(\pi R^3)$

Max stress occurs in hook if $|\sigma_{x\theta}^{hook}| > |\sigma_{x\theta}^{tail}|$, thus:

$$2 > 3(1-H/L) \implies \boxed{H > L/3}$$



Solutions: Problem 3

(a) Gas is pumped into the vessel to a gauge pressure $P > 0$. What is the pressurized length L_P of the cylinder in terms of known quantities?

$$\varepsilon_{\theta\theta} = 0 = \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{xx} + \sigma_{rr}^{\nearrow 0})] \implies \sigma_{\theta\theta} = \nu\sigma_{xx}$$

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{\theta\theta} + \sigma_{rr}^{\nearrow 0})] \implies \varepsilon_{xx} = \sigma_{xx}(1-\nu^2)$$

$$\sigma_{xx} = \frac{PR}{2t}, \quad \varepsilon_{xx} = \frac{L_P - L_0}{L_0} \implies \boxed{L_P = L_0\left[1 + \frac{PR}{2Et}(1-\nu^2)\right]}$$

(b) What is the tension F in the cables in terms of the final length L_F and known quantities?

$$\sum F_x = 0 = -F + \sigma_{xx}(2\pi R t)$$

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\cancelto{0}{\sigma_{\theta\theta}} + \cancelto{0}{\sigma_{rr}})] \qquad \Longrightarrow \qquad \sigma_{xx} = E\,\varepsilon_{xx}$$

$$\varepsilon_{xx} = \frac{L_F - L_0}{L_0} \qquad \Longrightarrow \qquad \boxed{F = 2\pi E R t \left(\frac{L_F - L_0}{L_0}\right)}$$

(c) What is the torque T acting on the deformable bar in terms of the tension F in the cables and known quantities?

$$\boxed{T = 2Fr}$$

(d) What is the twist angle ϕ of the deformable bar in terms of the applied torque T and known quantities?

$$\boxed{\phi = \frac{2Td}{\pi G r^4}}$$

(e) Write a compatibility equation relating the final length L_F, the pressurized length L_P, the deflection angle and known quantities.

$$\boxed{L_P - L_F = 2r\phi}$$

(f) Justify that these equations are enough to solve for any of the unknowns.

We have five linear equations (one from each part) in five unknowns (L_P, L_F, F, T, ϕ). Therefore, we have a complete system which we can solve.