2.003J SPRING 2011 QUIZ #1

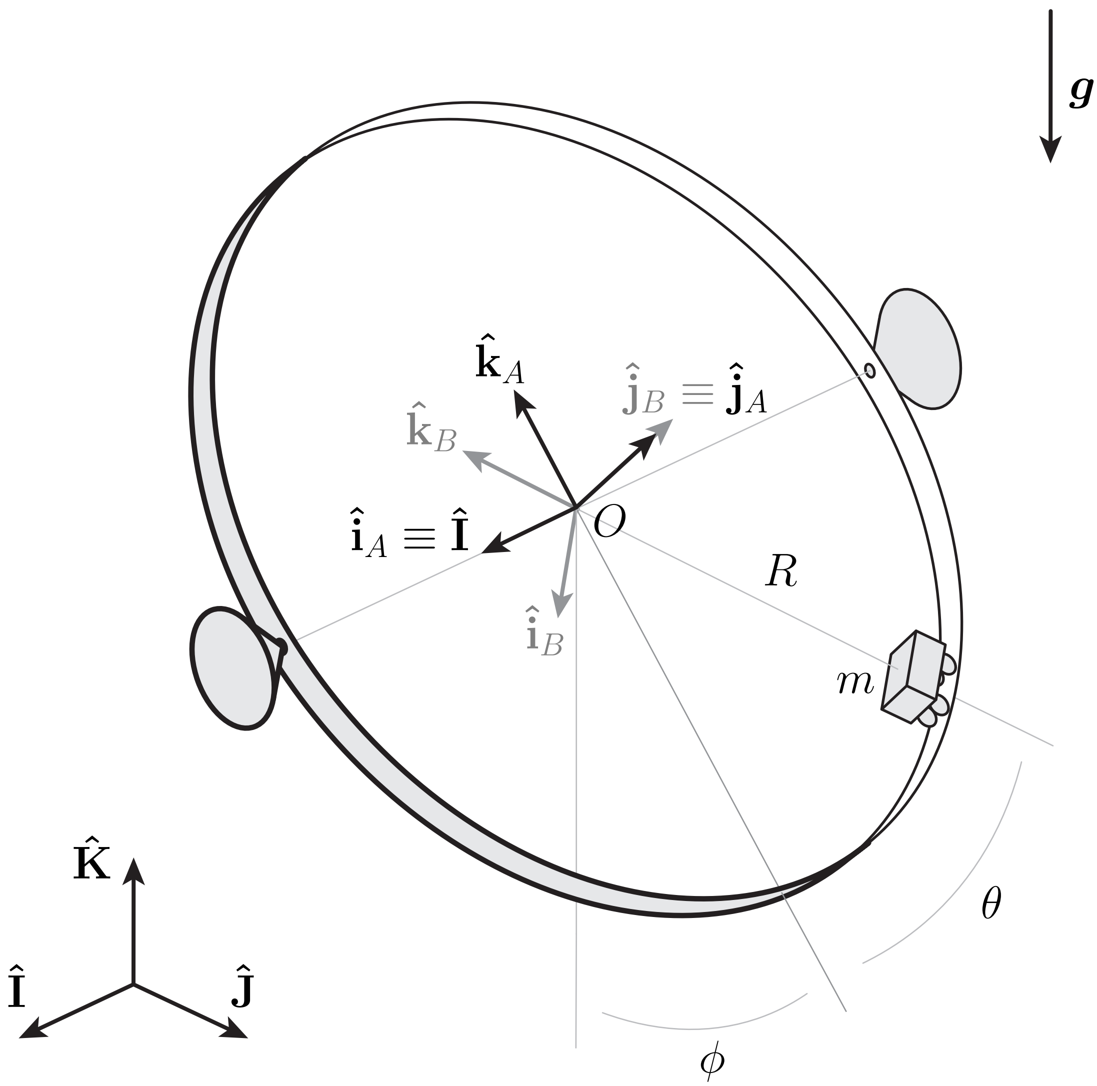
Wednesday, March 9th, 2011, 7:30pm-9:30pm
Walker Memorial Rm. 50-340

DO NOT TURN THIS PAGE OVER
UNTIL YOU ARE TOLD TO DO SO

- Be sure to WRITE YOUR NAME on your exam booklet.
- Please write all your work on the exam booklet provided. Only the work written in your exam booklet will be graded. Please draw a box around all final answers.
- Unless otherwise specified, feel free to express vector answers in terms of any unit coordinate vectors defined in the problem.
- This quiz has ten problems worth 10 points each. An eleventh bonus question marked (*B*) is extra credit and is worth an additional 10 points. You can receive partial credit, so please show your work.
- This quiz is closed book. You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- You have 120 minutes to complete this quiz.

2.003J Spring 2011: Dynamics and Control I	**Quiz 1**
Massachusetts Institute of Technology	**7:30pm-9:30pm**
Department of Mechanical Engineering	**March 9th, 2011**

Toy Car on Rotating Track



A small toy car of mass m moves on a larger circular track with radius R, center O, and negligible mass. The massless wheels of the car rotate on frictionless bearings but do not slip at their point of contact with the track. Thus, the car is constrained to move only in a tangential direction along the track. The track itself can pivot about frictionless bearings on an axis perpendicular to the direction of gravity $\boldsymbol{g}$. Assume the moment of inertia of the car is negligible such that it can be modeled as a point mass.

Define the fixed inertial reference frame $\hat{\text{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, frame $\hat{\text{A}} = (A \equiv O, \hat{\mathbf{i}}_A \equiv \hat{\mathbf{I}}, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A)$ rotating with the track, and frame $\hat{\text{B}} = (B \equiv O, \hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B \equiv \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_B)$ rotating with the car. Define ϕ to be the angle the track makes with the vertical ($\hat{\mathbf{K}} \times \hat{\mathbf{k}}_A = \sin\phi\,\hat{\mathbf{I}}$), and θ to be the angle the car makes from the bottom of the track ($\hat{\mathbf{k}}_A \times \hat{\mathbf{k}}_B = \sin\theta\,\hat{\mathbf{j}}_A$). These two variables ϕ and θ fully parameterize all possible configurations of the car. Please answer the following questions about this system.

Reference Frames and Initial Conditions

(1) Write down the unit vectors $\hat{\mathbf{i}}_B$, $\hat{\mathbf{j}}_B$, and $\hat{\mathbf{k}}_B$ in terms of θ, ϕ, and the ground unit vectors.

(2) Write down ${}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}}$, ${}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}$, and ${}^{\mathrm{B}}\boldsymbol{r}_m$ in terms of θ, ϕ, their derivatives, R, and any appropriate unit vectors. At time $t = 0$, the car has initial velocity $\boldsymbol{v}_0 = v_N\,\hat{\mathbf{j}}_B - v_T\,\hat{\mathbf{i}}_B$. Also write down the initial values of $\dot{\theta}_0$ and $\dot{\phi}_0$ in terms of v_N, v_T, R, and θ.

Momentum and Kinematics

(3) Write down the linear momentum ${}^{\mathrm{O}}\boldsymbol{p}_m$ of the car with respect to the ground reference frame in terms of θ, ϕ, their derivatives, R, m, and any appropriate unit vectors.

(4) Write down the angular momentum ${}^{O}\boldsymbol{h}_m^{O}$ of the car about point O with respect to the ground reference frame in terms of θ, ϕ, their derivatives, R, m, and any appropriate unit vectors.

(5) Write down the acceleration ${}^{\mathrm{O}}\boldsymbol{a}_m$ of the car with respect to the ground reference frame in terms of θ, ϕ, their derivatives, R, and any appropriate unit vectors. If there exists a non-zero Coriolis term, underline it. (Hint: $\hat{\mathbf{I}} \times \hat{\mathbf{i}}_B = \sin\theta\,\hat{\mathbf{j}}_B$, $\hat{\mathbf{I}} \times \hat{\mathbf{k}}_B = -\cos\theta\,\hat{\mathbf{j}}_B$, and $\hat{\mathbf{k}}_A \times \hat{\mathbf{k}}_B = \sin\theta\,\hat{\mathbf{j}}_B$)

Forces, Torques, and Equations of Motion

(6) The car's massless wheels rotate on their bearings without friction, thus the reaction force between the track and the car cannot point in the $\hat{\mathbf{i}}_B$ direction. Prove that the reaction force between the track and the car can only point in the $\hat{\mathbf{k}}_B$ direction, by considering an appropriate component of the angular momentum formulation of Newton's Second Law applied to the massless track.

(7) Draw a free-body diagram for the forces acting on the car. Write down each force in terms of any appropriate unit vectors.

(8) Write down the linear momentum formulation of Newton's Second Law for the sum of the forces acting on the car in a single vector equation in terms of any appropriate unit vectors.

(9) Write down the equations of motion for this system in terms of θ, ϕ, their derivatives, R, and g. These should be a system of two coupled scalar differential equations. (Hint: $\hat{\mathbf{k}}_A = \cos\theta\,\hat{\mathbf{k}}_B - \sin\theta\,\hat{\mathbf{i}}_B$ and $\hat{\mathbf{K}} = \cos\phi\cos\theta\,\hat{\mathbf{k}}_B - \cos\phi\sin\theta\,\hat{\mathbf{i}}_B + \sin\phi\,\hat{\mathbf{j}}_B$)

(10) Write down the reaction force $\boldsymbol{F}$ that the track must exert on the car as a function of θ, ϕ, their derivatives, R, m, g, and any appropriate unit vectors.

(*B*) Show that using the angular momentum formulation of Newton's Second Law of the car about point O results in the same equations of motion.

2.003J Spring 2011: Dynamics and Control I	**Quiz 1**
Massachusetts Institute of Technology	**7:30pm-9:30pm**
Department of Mechanical Engineering	**March 9th, 2011**

Solutions

(1) Write down the unit vectors $\hat{\mathbf{i}}_B$, $\hat{\mathbf{j}}_B$, and $\hat{\mathbf{k}}_B$ in terms of θ, ϕ, and the ground unit vectors.

$\boxed{\hat{\mathbf{i}}_B = \cos\theta\,\hat{\mathbf{I}} + \sin\theta\sin\phi\,\hat{\mathbf{J}} - \sin\theta\cos\phi\,\hat{\mathbf{K}}}$ $\boxed{\hat{\mathbf{j}}_B = \cos\phi\,\hat{\mathbf{J}} + \sin\phi\,\hat{\mathbf{K}}}$ $\boxed{\hat{\mathbf{k}}_B = \sin\theta\,\hat{\mathbf{I}} - \cos\theta\sin\phi\,\hat{\mathbf{J}} + \cos\theta\cos\phi\,\hat{\mathbf{K}}}$

These can be written by inspection or by using rotation matrices:

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta\sin\phi & -\sin\theta\cos\phi \\ 0 & \cos\phi & \sin\phi \\ \sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}_B \\ \hat{\mathbf{j}}_B \\ \hat{\mathbf{k}}_B \end{bmatrix}$$

(2) Write down ${}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}}$, ${}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}$, and ${}^{\mathrm{B}}\boldsymbol{r}_m$ in terms of θ, ϕ, their derivatives, R, and appropriate unit vectors. At time $t = 0$, the car has initial velocity $\boldsymbol{v}_0 = v_N\,\hat{\mathbf{j}}_B - v_T\,\hat{\mathbf{i}}_B$. Also write down the <u>initial values</u> of $\dot{\theta}_0$ and $\dot{\phi}_0$ in terms of v_N, v_T, R, and θ.

$$\boxed{{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} = \dot{\phi}\,\hat{\mathbf{I}}} \qquad \boxed{{}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} = \dot{\theta}\,\hat{\mathbf{j}}_A} \qquad \boxed{{}^{\mathrm{B}}\boldsymbol{r}_m = -R\,\hat{\mathbf{k}}_B}$$

$$\boxed{\dot{\theta}_0 = \frac{v_T}{R}} \qquad \boxed{\dot{\phi}_0 = \frac{v_N}{R\cos\theta}}$$

(3) Write down the linear momentum ${}^{\mathrm{O}}\boldsymbol{p}_m$ of the car with respect to the ground reference frame in terms of θ, ϕ, their derivatives, R, m, and any appropriate unit vectors.

$$\begin{aligned} {}^{\mathrm{O}}\boldsymbol{p}_m = m\,{}^{\mathrm{O}}\boldsymbol{v}_m = m\frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{O}}\boldsymbol{r}_m = m\frac{{}^{\mathrm{O}}d}{dt}\left(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{r}_B} + {}^{\mathrm{B}}\boldsymbol{r}_m\right) &= m\left(\cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_m} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_m\right) = m({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}) \times {}^{\mathrm{B}}\boldsymbol{r}_m \\ &= m(\dot{\phi}\,\hat{\mathbf{I}} + \dot{\theta}\,\hat{\mathbf{j}}_A) \times (-R\,\hat{\mathbf{k}}_B) = \boxed{mR(\dot{\phi}\cos\theta\,\hat{\mathbf{j}}_B - \dot{\theta}\,\hat{\mathbf{i}}_B)} \end{aligned}$$

(4) Write down the angular momentum ${}^{O}\boldsymbol{h}_m^{O}$ of the car about point O with respect to the ground reference frame in terms of θ, ϕ, their derivatives, R, m, and any appropriate unit vectors.

$${}^{O}\boldsymbol{h}_m^{O} = {}^{\mathrm{O}}\boldsymbol{r}_m \times {}^{\mathrm{O}}\boldsymbol{p}_m = (-R\,\hat{\mathbf{k}}_B) \times mR(\dot{\phi}\cos\theta\,\hat{\mathbf{j}}_B - \dot{\theta}\,\hat{\mathbf{i}}_B) = \boxed{mR^2(\dot{\phi}\cos\theta\,\hat{\mathbf{i}}_B + \dot{\theta}\,\hat{\mathbf{j}}_B)}$$

(5) Write down the acceleration ${}^{\mathrm{O}}\boldsymbol{a}_m$ of the car with respect to the ground reference frame in terms of θ, ϕ, their derivatives, R, and any appropriate unit vectors. If there exists a non-zero Coriolis term, underline it. (Hint: $\hat{\mathbf{I}} \times \hat{\mathbf{i}}_B = \sin\theta\,\hat{\mathbf{j}}_B$, $\hat{\mathbf{I}} \times \hat{\mathbf{k}}_B = -\cos\theta\,\hat{\mathbf{j}}_B$, and $\hat{\mathbf{k}}_A \times \hat{\mathbf{k}}_B = \sin\theta\,\hat{\mathbf{j}}_B$)

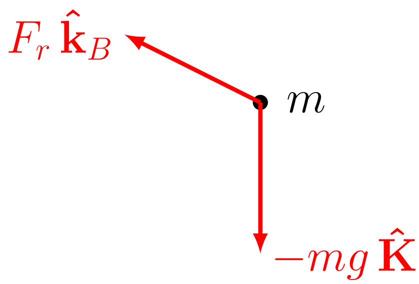
$$
\begin{aligned}
{}^{\mathrm{O}}\boldsymbol{a}_m &= \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{r}_m\right)\right) = \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}\left(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{r}_B} + {}^{\mathrm{B}}\boldsymbol{r}_m\right)\right) \\
&= \frac{{}^{\mathrm{O}}d}{dt}\left(\cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_m} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_m\right) = \frac{{}^{\mathrm{O}}d}{dt}\left(\left({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}\right) \times {}^{\mathrm{B}}\boldsymbol{r}_m\right) \\
&= \left({}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}} + {}^{\mathrm{A}}\dot{\boldsymbol{\omega}}_{\mathrm{B}} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}\right) \times {}^{\mathrm{B}}\boldsymbol{r}_m + \left({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + {}^{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}}\right) \times \left(\cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_m} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_m\right) \\
&= (\ddot{\phi}\,\hat{\mathbf{I}} + \ddot{\theta}\,\hat{\mathbf{j}}_A + \dot{\phi}\,\hat{\mathbf{I}} \times \dot{\theta}\,\hat{\mathbf{j}}_B) \times (-R\,\hat{\mathbf{k}}_B) + (\dot{\phi}\,\hat{\mathbf{I}} + \dot{\theta}\,\hat{\mathbf{j}}_B) \times \left((\dot{\phi}\,\hat{\mathbf{I}} + \dot{\theta}\,\hat{\mathbf{j}}_B) \times (-R\,\hat{\mathbf{k}}_B)\right) \\
&= R\ddot{\phi}\cos\theta\,\hat{\mathbf{j}}_B - R\ddot{\theta}\,\hat{\mathbf{i}}_B - R\dot{\phi}\dot{\theta}\sin\theta\,\hat{\mathbf{j}}_B + R\dot{\phi}^2\cos\theta\,\hat{\mathbf{k}}_A - R\dot{\phi}\dot{\theta}\sin\theta\,\hat{\mathbf{j}}_B - R\dot{\theta}^2\,\hat{\mathbf{k}}_B \\
&= \boxed{R\ddot{\phi}\cos\theta\,\hat{\mathbf{j}}_B - R\ddot{\theta}\,\hat{\mathbf{i}}_B - \underline{2R\dot{\phi}\dot{\theta}\sin\theta\,\hat{\mathbf{j}}_B} + R\dot{\phi}^2\cos\theta\,\hat{\mathbf{k}}_A + R\dot{\theta}^2\,\hat{\mathbf{k}}_B}
\end{aligned}
$$

(6) The car's massless wheels rotate on their bearings without friction, thus the reaction force between the track and the car cannot point in the $\hat{\mathbf{i}}_B$ direction. Prove that the reaction force between the track and the car can only point in the $\hat{\mathbf{k}}_B$ direction, by considering an appropriate component of the angular momentum formulation of Newton's Second Law applied to the massless track.

$$
\begin{aligned}
\sum \boldsymbol{\tau}_T^O &= \frac{{}^{\mathrm{O}}d}{dt}\left(\cancelto{0}{{}^{O}\boldsymbol{h}_T^O}\right) + \cancelto{0}{{}^{\mathrm{O}}\boldsymbol{v}_O} \times \cancelto{0}{{}^{\mathrm{O}}\boldsymbol{p}_T} = \mathbf{0} \\
\sum \boldsymbol{\tau}_T^O \cdot \hat{\mathbf{I}} &= \left({}^{\mathrm{O}}\boldsymbol{r}_m \times \boldsymbol{F}_r\right) \cdot \hat{\mathbf{I}} = \left[(-R\,\hat{\mathbf{k}}_B) \times \boldsymbol{F}_r\right] \cdot \hat{\mathbf{I}} = \mathbf{0}
\end{aligned}
$$

For this to always hold true, $\boldsymbol{F}_r$ must only point in the $\pm\,\hat{\mathbf{k}}_B$ direction. ✓

(7) Draw a free-body diagram for the forces acting on the car. Write down each force in terms of any appropriate unit vectors.



(8) Write down the linear momentum formulation of Newton's Second Law for the sum of the forces acting on the car in a single vector equation in terms of any appropriate unit vectors.

$$\sum \boldsymbol{F}_m = \frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{O}}\boldsymbol{p}_m \qquad F_r\,\hat{\mathbf{k}}_B - mg\,\hat{\mathbf{K}} = m{}^{\mathrm{O}}\boldsymbol{a}_m$$

$$\boxed{F_r\,\hat{\mathbf{k}}_B - mg\,\hat{\mathbf{K}} = mR(\ddot{\phi}\cos\theta\,\hat{\mathbf{j}}_B - \ddot{\theta}\,\hat{\mathbf{i}}_B - 2\dot{\phi}\dot{\theta}\sin\theta\,\hat{\mathbf{j}}_B + \dot{\phi}^2\cos\theta\,\hat{\mathbf{k}}_A + \dot{\theta}^2\,\hat{\mathbf{k}}_B)}$$

(9) Write down the equations of motion for this system in terms of θ, ϕ, their derivatives, R, and g. These should be a system of two coupled scalar differential equations. (Hint: $\hat{\mathbf{k}}_A = \cos\theta\,\hat{\mathbf{k}}_B - \sin\theta\,\hat{\mathbf{i}}_B$ and $\hat{\mathbf{K}} = \cos\phi\cos\theta\,\hat{\mathbf{k}}_B - \cos\phi\sin\theta\,\hat{\mathbf{i}}_B + \sin\phi\,\hat{\mathbf{j}}_B$)

$$F_r\,\hat{\mathbf{k}}_B - mg\,\hat{\mathbf{K}} = mR(\ddot{\phi}\cos\theta\,\hat{\mathbf{j}}_B - \ddot{\theta}\,\hat{\mathbf{i}}_B + 2\dot{\phi}\dot{\theta}\sin\theta\,\hat{\mathbf{j}}_B + \dot{\phi}^2\cos\theta\,\hat{\mathbf{k}}_A + \dot{\theta}^2\,\hat{\mathbf{k}}_B)$$

Note that all but two unit vectors, $\hat{\mathbf{K}}$ and $\hat{\mathbf{k}}_A$, are expressed in the $\hat{\mathrm{B}}$ reference frame. F_r only appears in the $\hat{\mathbf{k}}_B$ direction, so the equations in $\hat{\mathbf{i}}_B$ and $\hat{\mathbf{j}}_B$ will yield the equations of motion:

$$\hat{\mathbf{i}}_B : \boxed{0 = R\ddot{\theta} + R\dot{\phi}^2\cos\theta\sin\theta + g\cos\phi\sin\theta}$$

$$\hat{\mathbf{j}}_B : \boxed{0 = R\ddot{\phi}\cos\theta - 2R\dot{\phi}\dot{\theta}\sin\theta + g\sin\phi}$$

(10) Write down the reaction force $\boldsymbol{F}$ that the track must exert on the car as a function of θ, ϕ, their derivatives, R, m, g, and any appropriate unit vectors.

Projecting in the $\hat{\mathbf{k}}_B$ direction yields:

$$\boxed{\boldsymbol{F} = F_r\,\hat{\mathbf{k}}_B = m(g\cos\phi\cos\theta + R\dot{\phi}^2\cos^2\theta + R\dot{\theta}^2)\,\hat{\mathbf{k}}_B}$$

(*B*) Show that using the angular momentum formulation of Newton's Second Law of the car about point O results in the same equations of motion.

$$\begin{aligned}
\sum \boldsymbol{\tau}_m^O &= {}^{\mathrm{O}}\boldsymbol{r}_m \times (\boldsymbol{F} + \boldsymbol{F}_g) = (-R\,\hat{\mathbf{k}}_B) \times (F_r\,\hat{\mathbf{k}}_B - mg\,\hat{\mathbf{K}}) \\
&= -mg(\sin\theta\,\hat{\mathbf{i}}_B + \cos\theta\sin\phi\,\hat{\mathbf{j}}_B) \\
&= \frac{{}^{\mathrm{O}}d}{dt}({}^{O}\boldsymbol{h}_m^O) + \cancelto{0}{{}^{\mathrm{O}}\boldsymbol{v}_O} \times {}^{\mathrm{O}}\boldsymbol{p}_m \\
&= \frac{{}^{\mathrm{O}}d}{dt}(mR^2(\dot\phi\cos\theta\,\hat{\mathbf{i}}_B + \dot\theta\,\hat{\mathbf{j}}_B)) \\
&= mR^2(\ddot\phi\cos\theta\,\hat{\mathbf{i}}_B - \dot\phi\dot\theta\sin\theta\,\hat{\mathbf{i}}_B + \dot\phi\cos\theta\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times \hat{\mathbf{i}}_B + \ddot\theta\,\hat{\mathbf{j}}_B + \dot\theta\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times \hat{\mathbf{j}}_B) \\
&= mR(\ddot\phi\cos\theta\,\hat{\mathbf{i}}_B - \dot\phi\dot\theta\sin\theta\,\hat{\mathbf{i}}_B + \dot\phi\cos\theta(\dot\phi\,\hat{\mathbf{I}} + \dot\theta\,\hat{\mathbf{j}}_B) \times \hat{\mathbf{i}}_B + \ddot\theta\,\hat{\mathbf{j}}_B + \dot\theta(\dot\phi\,\hat{\mathbf{I}} + \dot\theta\,\hat{\mathbf{j}}_B) \times \hat{\mathbf{j}}_B) \\
&= mR(\ddot\phi\cos\theta\,\hat{\mathbf{i}}_B - \dot\phi\dot\theta\sin\theta\,\hat{\mathbf{i}}_B + \dot\phi^2\cos\theta\sin\theta\,\hat{\mathbf{j}}_B - \dot\phi\dot\theta\cos\theta\,\hat{\mathbf{k}}_B + \ddot\theta\,\hat{\mathbf{j}}_B + \dot\phi\dot\theta\,\hat{\mathbf{k}}_A) \\
&= mR(\ddot\phi\cos\theta\,\hat{\mathbf{i}}_B - 2\dot\phi\dot\theta\sin\theta\,\hat{\mathbf{i}}_B + \dot\phi^2\cos\theta\sin\theta\,\hat{\mathbf{j}}_B + \ddot\theta\,\hat{\mathbf{j}}_B)
\end{aligned}$$

$$-mg(\sin\theta\,\hat{\mathbf{i}}_B + \cos\theta\sin\phi\,\hat{\mathbf{j}}_B) = mR(\ddot\phi\cos\theta\,\hat{\mathbf{i}}_B - 2\dot\phi\dot\theta\sin\theta\,\hat{\mathbf{i}}_B + \dot\phi^2\cos\theta\sin\theta\,\hat{\mathbf{j}}_B + \ddot\theta\,\hat{\mathbf{j}}_B)$$

$$\hat{\mathbf{i}}_B : \boxed{0 = R\ddot\phi\cos\theta - 2R\dot\phi\dot\theta\sin\theta + g\sin\phi}$$

$$\hat{\mathbf{j}}_B : \boxed{0 = R\ddot\theta + R\dot\phi^2\cos\theta\sin\theta + g\cos\phi\sin\theta}$$

2.003J SPRING 2011 QUIZ #2

Wednesday, April 13th, 2011, 7:30pm-9:30pm

Walker Memorial Rm. 50-340

DO NOT TURN THIS PAGE OVER
UNTIL YOU ARE TOLD TO DO SO

- **Name:** ______________________________

- Please write all your work on this exam booklet. Only the work written in the exam booklet will be graded. Please draw a box around all final answers.

- This quiz has 2 problems worth 50 points each. A bonus question marked (*B*) is extra credit and is worth an additional 10 points. You can receive partial credit, so please show your work.

- This quiz is closed book. You are allowed two, two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.

- You have 120 minutes to complete this quiz.

In the following generally applicable equations, C represents the position of the center of mass, B is an arbitrary point of rotation, Ô is the inertial ground frame, and Â is any arbitrary frame.

Direct Method for Rigid Bodies

Newton's Second Law: $$\sum \boldsymbol{F}_M = M\,{}^{\mathrm{O}}\boldsymbol{a}_C \quad \text{for} \quad {}^{\mathrm{O}}\boldsymbol{r}_C = \frac{1}{M}\iiint_V \rho({}^{\mathrm{O}}\boldsymbol{r}_V)\,{}^{\mathrm{O}}\boldsymbol{r}_V\,dV$$

Euler's Equations: $$\sum \boldsymbol{\tau}_M^B = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{O}\boldsymbol{H}_M^B\right) + M\,{}^{\mathrm{O}}\boldsymbol{v}_B \times {}^{\mathrm{O}}\boldsymbol{v}_C \quad \text{for} \quad {}^{O}\boldsymbol{H}_M^B = \bar{\mathbf{I}}_M^B\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} + M\,{}^{\mathrm{B}}\boldsymbol{r}_C \times {}^{\mathrm{O}}\boldsymbol{v}_B$$

Variational Method

Lagrange's Equations: $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \Xi_j \qquad \text{for} \qquad \mathcal{L} = T - V \quad \text{and} \quad \Xi_j = \sum_{i=1}^{N} \boldsymbol{F}_i^{nc} \cdot \frac{\partial}{\partial q_j}{}^{\mathrm{O}}\boldsymbol{r}_i$$

Moment of Inertia

Definition for ${}^{\mathrm{B}}\boldsymbol{r}_{dV} = x\,\hat{\mathbf{i}}_A + y\,\hat{\mathbf{j}}_A + z\,\hat{\mathbf{k}}_A$:

$$\bar{\mathbf{I}}_M^B = \iiint_V \rho({}^{\mathrm{B}}\boldsymbol{r}_{dV})\left[({}^{\mathrm{B}}\boldsymbol{r}_{dV}\cdot{}^{\mathrm{B}}\boldsymbol{r}_{dV})\mathbb{1}_3 - ({}^{\mathrm{B}}\boldsymbol{r}_{dV}\otimes{}^{\mathrm{B}}\boldsymbol{r}_{dV})\right]dV$$

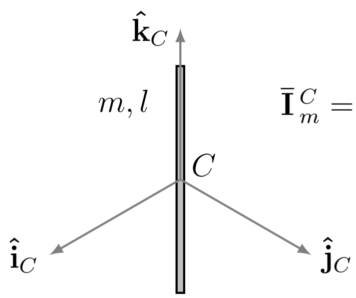
$$= \iiint_V \rho(x,y,z)\;{}^{A}\!\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & z^2+x^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} dx\,dy\,dz$$

Parallel Axis Theorem for ${}^{\mathrm{B}}\boldsymbol{r}_C = a\,\hat{\mathbf{i}}_A + b\,\hat{\mathbf{j}}_A + c\,\hat{\mathbf{k}}_A$:

$$\bar{\mathbf{I}}_M^B = \bar{\mathbf{I}}_M^C + M\left[({}^{\mathrm{B}}\boldsymbol{r}_C\cdot{}^{\mathrm{B}}\boldsymbol{r}_C)\bar{\mathbb{1}}_3 - ({}^{\mathrm{B}}\boldsymbol{r}_C\otimes{}^{\mathrm{B}}\boldsymbol{r}_C)\right] = \bar{\mathbf{I}}_M^C + M\;{}^{A}\!\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ba & c^2+a^2 & -bc \\ -ca & -cb & a^2+b^2 \end{bmatrix}$$

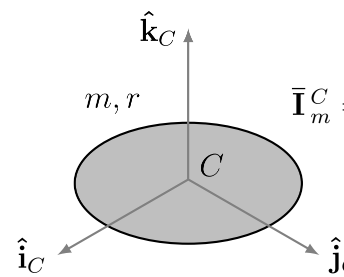
Useful moments of inertia about center of mass C with respect to frame $\hat{\mathrm{C}} = (C, \hat{\mathbf{i}}_C, \hat{\mathbf{j}}_C, \hat{\mathbf{k}}_C)$ oriented with the body's principal axes:

Thin Rod



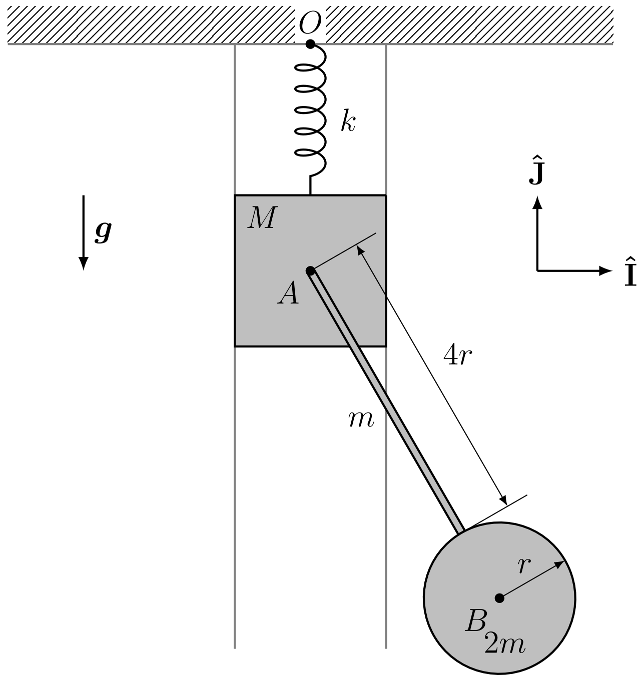
$$\bar{\mathbf{I}}_m^C = \tfrac{1}{12}ml^2\;{}^{C}\!\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thin Disk



$$\bar{\mathbf{I}}_m^C = \tfrac{1}{4}mr^2\;{}^{C}\!\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 1



A mass M is connected to a fixed ceiling by a spring with spring constant k. Mass M is constrained to move without friction in a vertical shaft strictly in the $\hat{\mathbf{J}}$ direction without rotation. Attached to mass M at point A is a solid pendulum made up of two rigid bodies: a rigid thin rod of mass m and length $4r$ rigidly attached to a thin, flat disk of mass $2m$ and radius r. Note that the pendulum cannot be modeled as a point mass. Assume the pivot point is frictionless and cannot transfer torque and that all motion is restricted to the plane. Also, let l_0 be the distance from point O to point A such that the spring is un-stretched.

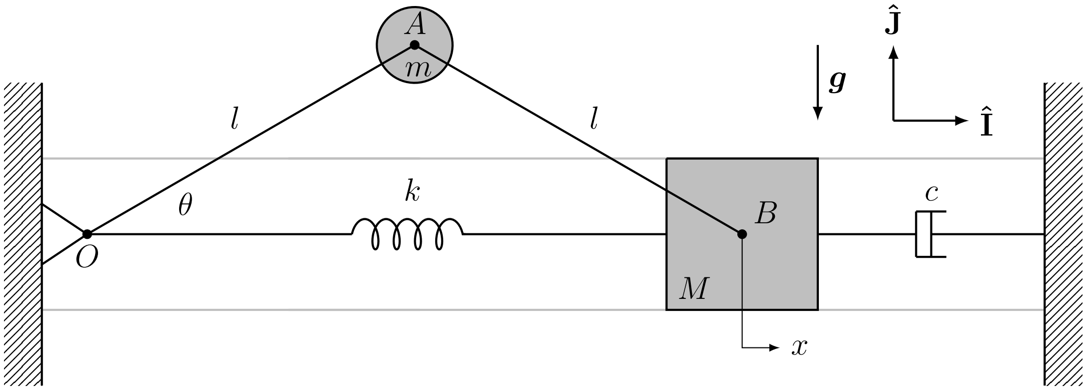
(a) Define a complete and independent set of scalar generalized coordinate(s) $(q_1, ..., q_n)$ that fully parameterize the state of the system. Note that n is the number of degrees of freedom of this system.

(b) Find the $\hat{\mathbf{K}} \otimes \hat{\mathbf{K}}$ component of the moment of inertia tensor $\bar{\mathbf{I}}_P^A \cdot (\hat{\mathbf{K}} \otimes \hat{\mathbf{K}})$ for the pendulum about point A.

(c) Find the equation(s) of motion for this system using the direct method.

(Each question is repeated in the following pages. Please write your answer on the appropriate page)

Problem 2



A mass M is connected to a fixed wall to the left by a spring with spring constant k, and to a fixed wall to the right by a dashpot with damping coefficient c. Mass M is constrained to move without friction in a horizontal shaft strictly in the $\hat{\mathbf{I}}$ direction without rotation. A point mass m is connected to points O and B via two rigid, inextensible massless rods, each of length l. Assume pivot points are frictionless and cannot transfer torque and that all motion is restricted to the plane. Also, let $l_0 < 2l$ be the distance from point O to point B such that the spring is un-stretched. For simplicity, assume that point mass m always remains above the line of the spring.

(a) How many degrees of freedom n does this system have? Define distance x as the distance B has moved from the spring's equilibrium position such that ${}^{\mathrm{O}}\boldsymbol{r}_B = (l_0 + x)\,\hat{\mathbf{I}}$. Also define angle θ as the angle between $\overline{OA}$ and $\overline{OB}$ such that ${}^{\mathrm{O}}\boldsymbol{r}_A = l(\cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}})$. Are x and θ independent variables? If they are not independent, determine how they depend on each other by finding $x = f(\theta)$, $\theta = f(x)$, $\dot{x} = f(\theta, \dot{\theta})$, and $\dot{\theta} = f(x, \dot{x})$.

(b) Find the equation(s) of motion for this system using the variational (indirect, Lagrangian) method.

(*B*) Solve for the force $\boldsymbol{F}_R$ that the right massless bar exerts on mass M. Express this force as a function of the generalized coordinate(s), expressed in terms of the ground unit vectors $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$.

(Each question is repeated in the following pages. Please write your answer on the appropriate page)

Problem 1

(a) Define a complete and independent set of scalar generalized coordinate(s) $(q_1, ..., q_n)$ that fully parameterize the motion of this system. Note that n is the number of degrees of freedom of this system.

This problem is a two degree of freedom problem, thus $n = 2$. Two natural variables that fully describe the motion of this system are a variable parameterizing the the linear displacement of vector ${}^{\mathrm{O}}\boldsymbol{r}_A$ and a variable parameterizing angular displacement of the vector ${}^{\mathrm{A}}\boldsymbol{r}_B$.

> Define distance y as the distance A has moved from the spring's equilibrium position such that ${}^{\mathrm{O}}\boldsymbol{r}_A = -(l_0+y)\,\hat{\mathbf{J}}$.

Additionally:

> Define angle θ as the angle between $\overline{AB}$ and the vertical such that ${}^{\mathrm{A}}\boldsymbol{r}_B = 5r(\sin\theta\,\hat{\mathbf{I}} - \cos\theta\,\hat{\mathbf{J}})$.

Problem 1

(b) Find the $\hat{\mathbf{K}} \otimes \hat{\mathbf{K}}$ component of the moment of inertia tensor $\bar{\mathbf{I}}_P^A \cdot (\hat{\mathbf{K}} \otimes \hat{\mathbf{K}})$ for the pendulum about point A.

Moment of inertia of pendulum about point A is given by tables and the parallel axis theorem:

$$\bar{\mathbf{I}}_P^A = \bar{\mathbf{I}}_{bar}^A + \bar{\mathbf{I}}_{disk}^A = \frac{16}{12}mr^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^A + m4r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^A + \frac{1}{4}2mr^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^A + 2m25r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^A$$

$$\bar{\mathbf{I}}_P^A = \left(\frac{331}{6}mr^2\right) \hat{\mathbf{i}}_A \otimes \hat{\mathbf{i}}_A + \left(\frac{1}{2}mr^2\right) \hat{\mathbf{j}}_A \otimes \hat{\mathbf{j}}_A + \left(\frac{169}{3}mr^2\right) \hat{\mathbf{k}}_A \otimes \hat{\mathbf{k}}_A = \frac{1}{6}mr^2 \begin{bmatrix} 331 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 338 \end{bmatrix}^A$$

$$\boxed{\bar{\mathbf{I}}_P^A \cdot (\hat{\mathbf{K}} \otimes \hat{\mathbf{K}}) = \frac{169}{3}mr^2}$$

Problem 1

(c) Find the equation(s) of motion for this system using the direct method.

Define ground reference frame $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$ and intermediate frame $\hat{\mathrm{A}} = (A, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A \equiv \hat{\mathbf{K}})$ which rotates with the pendulum such that ${}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} = \dot{\theta}\,\hat{\mathbf{K}}$ and ${}^{\mathrm{A}}\boldsymbol{r}_B = -5r\,\hat{\mathbf{j}}_A$. Taking Newton's second law for mass M:

$$\begin{aligned}\sum \boldsymbol{F}_M &= M\,{}^{\mathrm{O}}\boldsymbol{a}_A = M\frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{O}}\boldsymbol{r}_A\right) = M\frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}(-(l_0+y)\,\hat{\mathbf{J}})\right) = -M\ddot{y}\,\hat{\mathbf{J}}\\ &= F_S\,\hat{\mathbf{J}} - F_g\,\hat{\mathbf{J}} - F_N\,\hat{\mathbf{I}} + F_{Rx}\,\hat{\mathbf{I}} + F_{Ry}\,\hat{\mathbf{J}}\\ &= ky\,\hat{\mathbf{J}} - Mg\,\hat{\mathbf{J}} - F_N\,\hat{\mathbf{I}} + F_{Rx}\,\hat{\mathbf{I}} + F_{Ry}\,\hat{\mathbf{J}}\end{aligned}$$

Projecting in the $\hat{\mathbf{J}}$ direction yields:

$$\underline{M\ddot{y} + ky + F_{Ry} = Mg}$$

In order to apply Newton's second law to the pendulum, we must first find its center of mass C.

$${}^{\mathrm{A}}\boldsymbol{r}_C = \frac{1}{M_T}\iiint_V \rho({}^{\mathrm{A}}\boldsymbol{r}_{dV})\,{}^{\mathrm{A}}\boldsymbol{r}_{dV}\,dV = \frac{1}{3m}\left[m(-2r\,\hat{\mathbf{j}}_A) + 2m(-5r\,\hat{\mathbf{j}}_A)\right] = -4r\,\hat{\mathbf{j}}_A$$

Taking Newton's second law for the pendulum:

$$\begin{aligned}\sum \boldsymbol{F}_P &= 3m\,{}^{\mathrm{O}}\boldsymbol{a}_C = 3m\frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{O}}\boldsymbol{r}_A + {}^{\mathrm{A}}\boldsymbol{r}_C)\right) = 3m\frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}\left(-(l_0+y)\,\hat{\mathbf{J}} - 4r\,\hat{\mathbf{j}}_A\right)\right)\\ &= -3m\ddot{y}\,\hat{\mathbf{J}} - 12rm({}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}} \times \hat{\mathbf{j}}_A + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times \hat{\mathbf{j}}_A))\\ &= -3m\ddot{y}\,\hat{\mathbf{J}} + 12rm(\ddot{\theta}\,\hat{\mathbf{i}}_A + \dot{\theta}^2\,\hat{\mathbf{j}}_A)\\ &= -F_g\,\hat{\mathbf{J}} - F_{Rx}\,\hat{\mathbf{I}} - F_{Ry}\,\hat{\mathbf{J}}\\ &= -3mg\,\hat{\mathbf{J}} - F_{Rx}\,\hat{\mathbf{I}} - F_{Ry}\,\hat{\mathbf{J}}\end{aligned}$$

Projecting in the $\hat{\mathbf{J}}$ direction yields:

$$\underline{-3m\ddot{y} + 12rm(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = -3mg - F_{Ry}}$$

Combining underlined equations yields one equation of motion:

$$\boxed{(M+3m)\ddot{y} - 12rm(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) + ky = (M+3m)g}$$

Problem 1

(c) continued...

Taking Euler's Equations for the pendulum about point A:

$$\sum \boldsymbol{\tau}_P^A = \frac{{}^{\mathrm{O}}d}{dt}(\bar{\mathbf{I}}_P^{A\,\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}}) + 3m\left[\frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{A}}\boldsymbol{r}_C \times {}^{\mathrm{O}}\boldsymbol{v}_A) + {}^{\mathrm{O}}\boldsymbol{v}_A \times {}^{\mathrm{O}}\boldsymbol{v}_C\right]$$

Note that pivot point A is not the center of mass and is not stationary in the ground frame, thus the second term on the right will be non-zero. Only torque about A acting on the pendulum is due to gravity:

$$\sum \boldsymbol{\tau}_P^A = {}^{\mathrm{A}}\boldsymbol{r}_C \times \boldsymbol{F}_g = (-4r\,\hat{\mathbf{j}}_A) \times (-mg\,\hat{\mathbf{J}}) = -12rmg\sin\theta\,\hat{\mathbf{K}}$$

Thus:

$$-12rmg\sin\theta\,\hat{\mathbf{K}} = \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{169}{3}mr^2\dot{\theta}\,\hat{\mathbf{K}}\right) + 3m\left[\frac{{}^{\mathrm{O}}d}{dt}\left((-4r\,\hat{\mathbf{j}}_A)\times(-\dot{y}\,\hat{\mathbf{J}})\right) + (-\dot{y}\,\hat{\mathbf{J}})\times(-\dot{y}\,\hat{\mathbf{J}} + 4r\dot{\theta}\,\hat{\mathbf{i}}_A)\right]$$

$$= \frac{169}{3}mr^2\ddot{\theta}\,\hat{\mathbf{K}} + 3m\left[-4r(\ddot{y}\sin\theta + \dot{\theta}\dot{y}\cos\theta)\,\hat{\mathbf{K}} + 4r\dot{\theta}\dot{y}\cos\theta\,\hat{\mathbf{K}})\right]$$

$$\boxed{\frac{169}{3}mr^2\ddot{\theta} + 12mgr\sin\theta = 12mr\ddot{y}\sin\theta}$$

Note that the variational method would yield the same equations of motion. There are no non-conservative forces, thus all Ξ_i must be zero.

$$T = \frac{1}{2}M({}^{\mathrm{O}}\boldsymbol{v}_A\cdot{}^{\mathrm{O}}\boldsymbol{v}_A) + \frac{1}{2}m({}^{\mathrm{O}}\boldsymbol{v}_C\cdot{}^{\mathrm{O}}\boldsymbol{v}_C) + \frac{1}{2}{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}}\cdot(\bar{\mathbf{I}}_P^{C\,\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}}) = \frac{1}{2}M\dot{y}^2 + \frac{3}{2}m[(4r\dot{\theta}\sin\theta - \dot{y})^2 + 16r^2\dot{\theta}^2\cos^2\theta] + \frac{25}{6}mr^2\dot{\theta}^2$$

$$V = Mg({}^{\mathrm{O}}\boldsymbol{r}_A\cdot\hat{\mathbf{J}}) + 3mg({}^{\mathrm{O}}\boldsymbol{r}_C\cdot\hat{\mathbf{J}}) + \frac{1}{2}k({}^{\mathrm{E}}\boldsymbol{r}_A\cdot{}^{\mathrm{E}}\boldsymbol{r}_A) = -Mgy - 3mg(y + 4r\cos\theta) + \frac{1}{2}ky^2$$

$$\mathcal{L} = T - V = \frac{1}{2}M\dot{y}^2 + \frac{3}{2}m[(4r\dot{\theta}\sin\theta - \dot{y})^2 + 16r^2\dot{\theta}^2\cos^2\theta] + \frac{25}{6}mr^2\dot{\theta}^2 + Mgy + 3mg(y + 4r\cos\theta) - \frac{1}{2}ky^2$$

Problem 1

(c) continued...

For $q_1 = y$:

Lagrange's Equations: $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = \Xi_1 = 0$$

$$\frac{d}{dt}\Big(M\dot{y} - 3m(4r\dot{\theta}\sin\theta - \dot{y})\Big) - [(M+3m)g - ky] = 0$$

$$\boxed{(M+3m)\ddot{y} - 12mr(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) + ky = (M+3m)g}$$

For $q_2 = \theta$:

Lagrange's Equations: $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \Xi_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(48mr^2\dot{\theta} - 12mr\dot{y}\sin\theta + \frac{25}{6}mr^2\dot{\theta}\right)$$
$$= 48mr^2\ddot{\theta} - 12mr\ddot{y}\sin\theta - 12mr\dot{\theta}\dot{y}\cos\theta + \frac{25}{3}mr^2\ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 3m(4r\dot{\theta}\sin\theta - \dot{y})4r\dot{\theta}\cos\theta - 3m(16r^2\dot{\theta}^2\cos\theta\sin\theta) - 12mgr\sin\theta$$
$$= -12mr\dot{\theta}\dot{y}\cos\theta - 12mgr\sin\theta$$

$$\boxed{\frac{169}{3}mr^2\ddot{\theta} + 12mgr\sin\theta = 12mr\ddot{y}\sin\theta}$$

Problem 2

(a) How many degrees of freedom n does this system have? Define distance x as the distance B has moved from the spring's equilibrium position such that ${}^{\mathrm{O}}\boldsymbol{r}_B = (l_0 + x)\,\hat{\mathbf{I}}$. Also define angle θ as the angle between $\overline{OA}$ and $\overline{OB}$ such that ${}^{\mathrm{O}}\boldsymbol{r}_A = l(\cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}})$. Are x and θ independent variables? If they are not independent, determine how they depend on each other by finding $x = f(\theta)$, $\theta = f(x)$, $\dot{x} = f(\theta, \dot{\theta})$, and $\dot{\theta} = f(x, \dot{x})$.

This problem is a single degree of freedom problem, so $\boxed{n = 1}$. Thus, x and θ are $\boxed{\text{not independent}}$ and are geometrically related in the following way:

$$l_0 + x = 2l\cos\theta \qquad \text{so} \qquad \boxed{x = 2l\cos\theta - l_0} \qquad \boxed{\theta = \arccos\left(\frac{l_0 + x}{2l}\right)}$$

Taking derivatives yields:

$$\boxed{\dot{x} = -2l\dot{\theta}\sin\theta} \qquad \dot{\theta} = \frac{-\dot{x}}{2l\sin\theta} = \frac{-\dot{x}}{2l}\frac{2l}{\sqrt{4l^2 - (l_0 + x)^2}} \qquad \text{so} \qquad \boxed{\dot{\theta} = \frac{-\dot{x}}{\sqrt{4l^2 - (l_0 + x)^2}}}$$

Though it was not asked, accelerations of x and θ can also be related by taking an additional derivative:

$$\ddot{x} = -2l\ddot{\theta}\sin\theta - 2l\dot{\theta}^2\cos\theta \qquad \ddot{\theta} = \frac{-\ddot{x}}{\sqrt{4l^2 - (l_0 + x)^2}} + \frac{-(l_0 + x)\dot{x}^2\sqrt{4l^2 - (l_0 + x)^2}}{(4l^2 - (l_0 + x)^2)^2}$$

Problem 2

(b) Find the equation(s) of motion for this system using the <u>variational</u> (indirect, Lagrangian) method.

Define ground reference frame $\hat{\text{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$. Write down what we know:

$$^{\text{O}}\boldsymbol{r}_A = l(\cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}}) \qquad ^{\text{O}}\boldsymbol{r}_B = (l_0 + x)\,\hat{\mathbf{I}}$$

$$^{\text{O}}\boldsymbol{v}_A = \frac{^{\text{O}}d}{dt}(^{\text{O}}\boldsymbol{r}_A) = \frac{^{\text{O}}d}{dt}[l(\cos\theta\,\hat{\mathbf{I}} + l\sin\theta\,\hat{\mathbf{J}})] = l\dot{\theta}(-\sin\theta\,\hat{\mathbf{I}} + \cos\theta\,\hat{\mathbf{J}}) \qquad ^{\text{O}}\boldsymbol{v}_B = \frac{^{\text{O}}d}{dt}(^{\text{O}}\boldsymbol{r}_B) = \frac{^{\text{O}}d}{dt}[(l_0+x)\,\hat{\mathbf{I}}] = \dot{x}\,\hat{\mathbf{I}}$$

$$T = \frac{1}{2}m(^{\text{O}}\boldsymbol{v}_A \cdot {}^{\text{O}}\boldsymbol{v}_A) + \frac{1}{2}M(^{\text{O}}\boldsymbol{v}_B \cdot {}^{\text{O}}\boldsymbol{v}_B) = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2$$

$$V = mg(^{\text{O}}\boldsymbol{r}_A \cdot \hat{\mathbf{J}}) + \frac{1}{2}k(^{\text{O}}\boldsymbol{r}_B \cdot {}^{\text{O}}\boldsymbol{r}_B) = mgl\sin\theta + \frac{1}{2}kx^2$$

$$\mathcal{L} = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 - mgl\sin\theta - \frac{1}{2}kx^2$$

For $\quad q_1 = x:$

All forces acting on the system are conservative except for the force from the dashpot which removes energy from the system. Thus:

$$\Xi_1 = \sum_{i=1}^{N} \boldsymbol{F}_i^{nc} \cdot \frac{\partial}{\partial q_1}{}^{\text{O}}\boldsymbol{r}_i = \boldsymbol{F}_D \cdot \frac{^{O}\partial}{\partial q_1}{}^{\text{O}}\boldsymbol{r}_B = (-c\dot{x}\,\hat{\mathbf{I}}) \cdot \frac{\partial}{\partial x}[(l_0 + x)\,\hat{\mathbf{I}}] = -c\dot{x}$$

$$\mathcal{L} = \frac{1}{2}ml^2\frac{\dot{x}^2}{4l^2 - (l_0 + x)^2} + \frac{1}{2}M\dot{x}^2 - mgl\frac{\sqrt{4l^2 - (l_0 + x)^2}}{2l} - \frac{1}{2}kx^2$$

Plugging into Lagrange's Equations: $\quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = \Xi_1$

$$\frac{d}{dt}\left(ml^2\frac{\dot{x}}{4l^2 - (l_0 + x)^2} + M\dot{x}\right) - \left(ml^2\frac{(l_0 + x)\dot{x}^2}{(4l^2 - (l_0 + x)^2)^2} + mg\frac{l_0 - x}{\sqrt{4l^2 - (l_0 + x)^2}} - kx\right) = -c\dot{x}$$

$$\boxed{M\ddot{x} + ml^2\left(\frac{\ddot{x}}{4l^2 - (l_0 + x)^2} + \frac{(l_0 + x)\dot{x}^2}{(4l^2 - (l_0 + x)^2)^2}\right) + c\dot{x} + kx = mg\frac{l_0 - x}{\sqrt{4l^2 - (l_0 + x)^2}}}$$

Problem 2

(b) continued...

For $q_1 = \theta$:

All forces acting on the system are conservative except for the force from the dashpot which removes energy from the system. Thus:

$$\Xi_1 = \sum_{i=1}^{N} \boldsymbol{F}_i^{nc} \cdot \frac{\partial}{\partial q_1} {}^{\mathrm{O}}\boldsymbol{r}_i = \boldsymbol{F}_D \cdot \frac{\partial}{\partial q_1} {}^{\mathrm{O}}\boldsymbol{r}_B = (2cl\dot{\theta}\sin\theta\,\hat{\mathbf{I}}) \cdot \frac{\partial}{\partial\theta}(2l\cos\theta\,\hat{\mathbf{I}}) = -4cl^2\dot{\theta}\sin^2\theta$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta - mgl\sin\theta - \frac{1}{2}k(2l\cos\theta - l_0)^2$$

Plugging into Lagrange's Equations: $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \Xi_1$

$$\frac{d}{dt}\left(ml^2\dot{\theta} + 4Ml^2\dot{\theta}\sin^2\theta\right) - \left[4Ml^2\dot{\theta}^2\sin\theta\cos\theta - mgl\cos\theta + k(2l\cos\theta - l_0)2l\sin\theta\right] = -4cl^2\dot{\theta}\sin^2\theta$$

$$\boxed{ml^2\ddot{\theta} + 4Ml^2(\ddot{\theta}\sin^2\theta + \dot{\theta}^2\sin\theta\cos\theta) + 4cl^2\dot{\theta}\sin^2\theta + mgl\cos\theta = 2kl(2l\cos\theta - l_0)\sin\theta}$$

Problem 2

(*B*) Solve for the force $\boldsymbol{F}_R$ that the right massless bar exerts on mass M. Express this force as a function of the generalized coordinate(s), expressed in terms of the ground unit vectors $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$.

The direct method can help us solve for internal reaction forces. Note that because the rigid rods are massless, the sum of the torques on them must be zero, and, since no torques act at either pivot the reaction forces at either end must be collinear with the rod. Taking Newton's second law for mass M:

$$\begin{aligned}\sum \boldsymbol{F}_M &= M\,{}^{\mathrm{O}}\boldsymbol{a}_B = M\frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{O}}\boldsymbol{r}_B\right) = M\frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}(l_0+x)\,\hat{\mathbf{I}}\right) = M\ddot{x}\,\hat{\mathbf{I}} \\ &= F_N\,\hat{\mathbf{J}} - F_g\,\hat{\mathbf{J}} - F_S\,\hat{\mathbf{I}} - F_D\,\hat{\mathbf{I}} + \boldsymbol{F}_R \\ &= F_N\,\hat{\mathbf{J}} - F_g\,\hat{\mathbf{J}} - kx\,\hat{\mathbf{I}} - c\dot{x}\,\hat{\mathbf{I}} + F_R(\cos\theta\,\hat{\mathbf{I}} - \sin\theta\,\hat{\mathbf{J}})\end{aligned}$$

Projecting in the $\hat{\mathbf{I}}$ direction yields:

$$F_R\cos\theta = M\ddot{x} + c\dot{x} + kx$$

Thus:

$$\boldsymbol{F}_R = (M\ddot{x} + c\dot{x} + kx)(\hat{\mathbf{I}} - \tan\theta\,\hat{\mathbf{J}})$$

For $q_1 = x$:

$$\boxed{\boldsymbol{F}_R = (M\ddot{x} + c\dot{x} + kx)\left(\hat{\mathbf{I}} - \frac{\sqrt{4l^2 - (l_0+x)^2}}{l_0+x}\,\hat{\mathbf{J}}\right)}$$

For $q_1 = \theta$:

$$\boxed{\boldsymbol{F}_R = [M(-2l\ddot{\theta}\sin\theta - 2l\dot{\theta}^2\cos\theta) + c(-2l\dot{\theta}\sin\theta) + k(2l\cos\theta)]\left(\hat{\mathbf{I}} - \tan\theta\,\hat{\mathbf{J}}\right)}$$

2.003J SPRING 2011 FINAL EXAM

Monday, May 16th, 2011, 9:00am-12:00pm

Johnson Ice Rink

DO NOT TURN THIS PAGE OVER
UNTIL YOU ARE TOLD TO DO SO

- **Name:** ______________________________

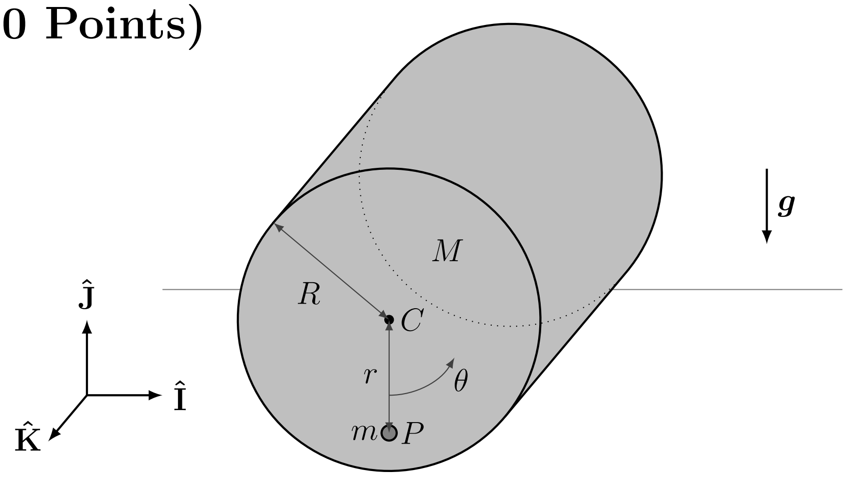
- Please write all your work on this exam booklet. Only the work written in the exam booklet will be graded. Please draw a box around all final answers.

- This quiz will be scored out of 100 points distributed over four problems. Point values are given for each part of each problem.

- This quiz is closed book and closed notes. You are allowed three, two-sided, handwritten, 8.5" by 11" formula sheets. Calculators are not allowed.

- You have 180 minutes to complete this Final Exam.

Problem 1 (30 Points)



A uniform cylinder of mass M, radius R and length L has a small point mass m attached to its circular base at a distance r from the center. The cylinder rolls without slip along a horizontal plane in the $\hat{\mathbf{I}}$ direction in the presence of gravity $-g\,\hat{\mathbf{J}}$.

(a) (10 points) Find the non-linear equation of motion of the system in terms of the angle of rotation θ.

(b) (10 points) Find the static equilibrium positions and analyze their stability.

(c) (10 points) Linearize the equation of motion around stable equilibria and determine the natural period of oscillation T_n.

(a) (10 points) Find the non-linear equation of motion of the system in terms of the angle of rotation θ.

Either the direct or indirect method can be used to solve for the equation of motion. Indirect yields:

$$L = T - V = \frac{1}{2}M\,{}^{O}\boldsymbol{v}_C \cdot {}^{O}\boldsymbol{v}_C + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\,{}^{O}\boldsymbol{v}_P \cdot {}^{O}\boldsymbol{v}_P - m\,{}^{O}\boldsymbol{r}_P \cdot \boldsymbol{g}$$

$${}^{O}\boldsymbol{v}_C = -R\dot{\theta}\,\hat{\mathbf{I}} \qquad {}^{O}\boldsymbol{v}_P = -R\dot{\theta}\,\hat{\mathbf{I}} + r\dot{\theta}(\cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}}) \qquad {}^{O}\boldsymbol{r}_P = -R\theta\,\hat{\mathbf{I}} + R\,\hat{\mathbf{J}} + r(\sin\theta\,\hat{\mathbf{I}} - \cos\theta\,\hat{\mathbf{J}})$$

$$L = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}\left[\frac{1}{2}MR^2\right]\dot{\theta}^2 + \frac{1}{2}m\dot{\theta}^2[(R - r\cos\theta)^2 + r^2\sin^2\theta] - mg(R - r\cos\theta)$$

$$L = \left[\frac{3}{2}MR^2 + m(R^2 + r^2 + 2rR\cos\theta)\right]\frac{1}{2}\dot{\theta}^2 - mg(R - r\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \Xi_\theta = 0 \qquad \text{No non-conservative forces}$$

$$\boxed{\left[\frac{3}{2}MR^2 + m(R^2 + r^2 - 2rR\cos\theta)\right]\ddot{\theta} + mgr\sin\theta = 0}$$

(b) (10 points) Find the static equilibrium positions and analyze their stability.

$$\left[\frac{3}{2}MR^2 + m(R^2 + r^2 - 2rR\cos\theta)\right]\ddot{\theta} + mgr\sin\theta = 0$$

Static equilibrium when $\ddot{\theta} = \dot{\theta} = 0$, thus when $\sin\theta = 0$, yielding:

$$\boxed{\theta_{eq} = 0 + n\pi \qquad \forall\ n = \ldots, -2, -1, 0, 1, 2, \ldots}$$

For n even, the system is $\boxed{\text{stable}}$. For n odd, the system is $\boxed{\text{unstable}}$.

(c) (10 points) Linearize the equation of motion around stable equilibria and determine the natural period of oscillation T_n.

$$f(\theta)\ddot{\theta} + h(\theta) = 0 \qquad f(\theta) = \left[\frac{3}{2}MR^2 + m(R^2 + r^2 - 2rR\cos\theta)\right] \qquad h(\theta) = mgr\sin\theta$$

$$\theta_{eq} = 0 + n\pi \qquad \forall\ n = \ldots, -2, -1, 0, 1, 2, \ldots$$

Linearization around equilibria given by $f(\theta_{eq})\ddot{\theta} + h'(\theta eq) = 0$, thus for n even:

$$\boxed{\left[\frac{3}{2}MR^2 + m(R-r)^2\right]\ddot{\theta} + mgr\theta = 0}$$

while for n odd (unstable):

$$\left[\frac{3}{2}MR^2 + m(R+r)^2\right]\ddot{\theta} - mgr\theta = 0$$

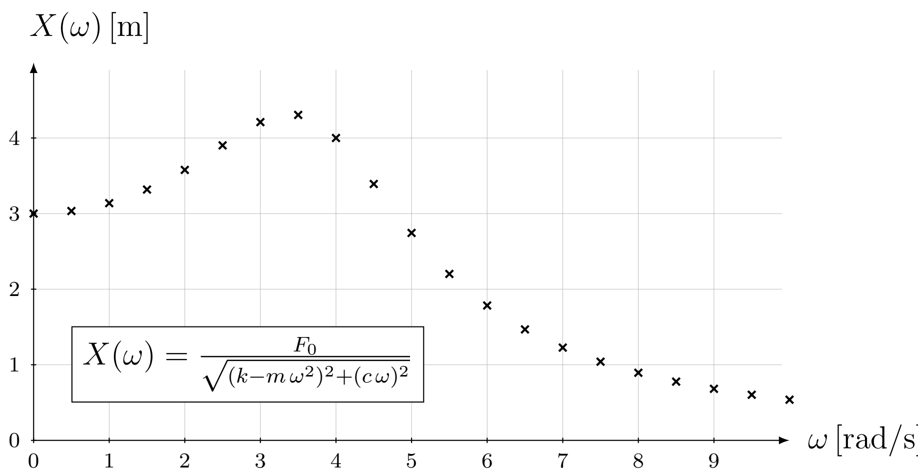
Natural frequency ω_n of stable system is:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{mgr}{\frac{3}{2}MR^2 + m(R-r)^2}} \qquad \boxed{T_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\frac{3}{2}MR^2 + m(R-r)^2}{mgr}}}$$

Using parameters $\sigma = R/r$ for $\sigma > 1$ and $\mu = M/m$, natural period becomes:

$$T_n = 2\pi\sqrt{\frac{r}{g}}\sqrt{(\sigma-1)^2 + \frac{3}{2}\mu\sigma^2}$$

Problem 2 (15 points)



We apply a known oscillatory force $F(t) = F_0 \cos(\omega t)$ with $F_0 = 9$ N to an unknown system at twenty-one different known frequencies ω from 0 to 10 rad/s. The amplitude of the steady state response of the system for each ω is plotted above. Assume the system behaves as a second order linear system of the form $m\ddot{x} + c\dot{x} + kx = F(t)$.

(a) (8 points) Estimate the physical values of m, c, and k for this system.

(b) (7 points) Write down and sketch the unforced response ($F_0 = 0$) of this system with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

(a) (8 points) Estimate the physical values of m, c, and k for this system.

$$X(0 \text{ rad/s}) = 3 \text{ m} = \frac{F_0}{k} = \frac{9 \text{ N}}{k} \qquad \boxed{k = 3 \text{ N/m}}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3 \text{ N/m}}{m}} \approx 4 \text{ rad/s} \qquad \boxed{m \approx \frac{3}{16} \text{ kg}}$$

$$X(4 \text{ rad/s}) = 4 \text{ m} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \boxed{c \approx \frac{9}{16} \text{ kg/s}}$$

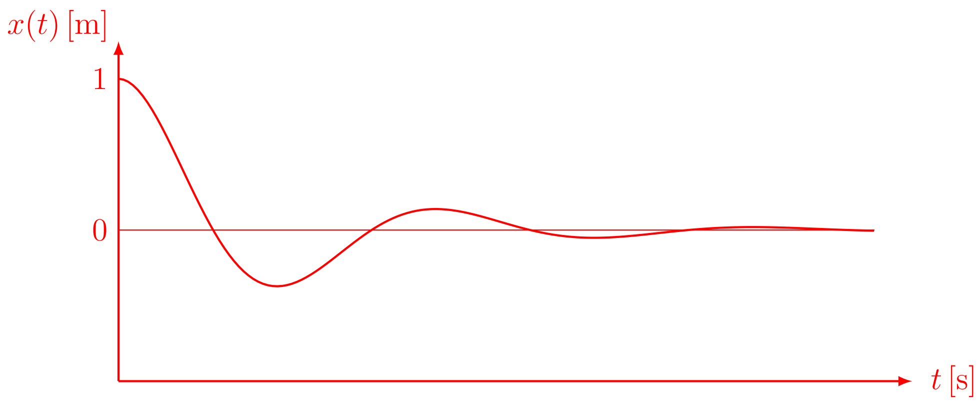
(b) (7 points) Write down and sketch the unforced response ($F_0 = 0$) of this system with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

$$\omega_n \approx 4 \text{ rad/s} \qquad \zeta = \frac{c}{2\sqrt{mk}} \approx \frac{3}{8}$$

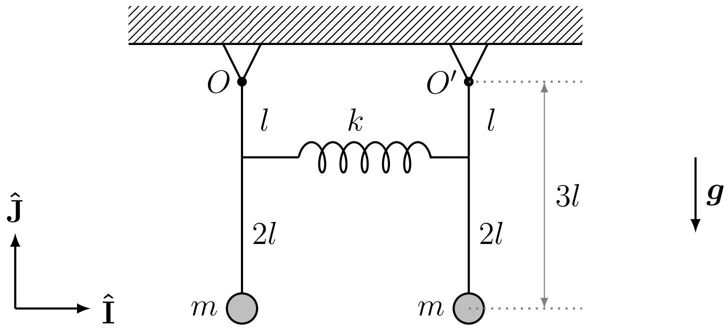
$$x(t) = e^{-\zeta\omega_n t}[A\sin(\omega_d t) + B\cos(\omega_d t)] \qquad \text{for} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$x(0) = 1 \qquad B = 1 \qquad \dot{x}(0) = 0 \qquad A = -\frac{\zeta}{\sqrt{1-\zeta}}$$

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left[\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta}}\sin(\omega_d t)\right] \qquad \text{for} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}}$$



Problem 3 (25 points)



Two identical pendulums made of massless rigid rods of length $3l$ and point masses of mass m are connected at a distance l from their frictionless pivots O and O' by a spring k such that the spring is un-stretched when both pendulums are vertical. Let θ_1 and θ_2 measure the angles of rotation of the left and right pendulums respectively in the counter-clockwise direction from the vertical positions shown.

(a) (10 points) Find the linear equations of motion for small motions of this coupled two-degree of freedom system in θ_1 and θ_2 for small deviations from $\theta_1 = \theta_2 = 0$. The solution will be of the form below. What are the values of the constants? (**Hint:** Assume for small motions that the spring force will be horizontal along the $\hat{\mathbf{I}}$ direction)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) (15 points) Solve for the natural modes of vibration for this system and the natural frequency associated with each mode.

(a) (10 points) Find the linear equations of motion for small motions of this coupled two-degree of freedom system in θ_1 and θ_2 for small deviations from $\theta_1 = \theta_2 = 0$. The solution will be of the form below. What are the values of the constants? (**Hint:** Assume for small motions that the spring force will be horizontal along the $\hat{\mathbf{I}}$ direction)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Euler's equation for each pendulum around its fixed point of rotation for small θ, so $\sin\theta \approx \theta$:

$$\sum \boldsymbol{\tau}^O = -kl(l\theta_1 - l\theta_2) - mg(3l\theta_1) = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{r}_m \times m\,{}^{\mathrm{O}}\boldsymbol{v}_m\right) = m(3l)(3l\ddot{\theta}_1)$$

$$\boxed{9ml\ddot{\theta}_1 + (kl + 3mg)\theta_1 - kl\theta_2 = 0}$$

$$\sum \boldsymbol{\tau}^{O'} = -kl(l\theta_2 - l\theta_1) - mg(3l\theta_2) = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{r}_m \times m\,{}^{\mathrm{O}}\boldsymbol{v}_m\right) = m(3l)(3l\ddot{\theta}_2)$$

$$\boxed{9ml\ddot{\theta}_2 + (kl + 3mg)\theta_2 - kl\theta_1 = 0}$$

$$\boxed{m_1 = m_2 = 9ml} \qquad \boxed{k_1 = k_4 = kl + 3mg} \qquad \boxed{k_2 = k_3 = -kl}$$

(b) (15 points) Solve for the natural modes of vibration for this system and the natural frequency associated with each mode.

Assume solution of the form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(\omega t + \phi)$$

$$\left(-\omega^2 \begin{bmatrix} m_1 a_1 \\ m_2 a_2 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) \sout{\sin(\omega t + \phi)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{a_1}{a_2} = \frac{-k_2}{k_1 - m_1\omega^2} = \frac{k_4 - m_2\omega^2}{-k_3}$$

$$k_2 k_3 = (k_1 - m_1\omega^2)(k_4 - m_2\omega^2)$$

$$m_1 m_2 \omega^4 - (m_1 k_4 + m_2 k_1)\omega^2 + (k_1 k_4 - k_2 k_3) = 0$$

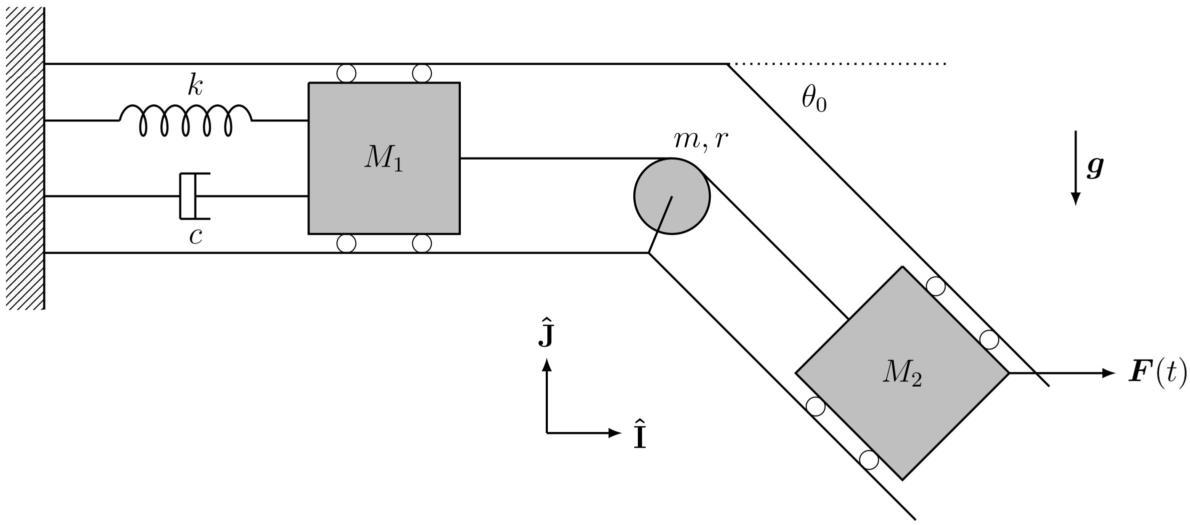
$$\omega^2 = \frac{(m_1 k_4 + m_2 k_1) \pm \sqrt{(m_1 k_4 + m_2 k_1)^2 - 4 m_1 m_2 (k_1 k_4 - k_2 k_3)}}{2 m_1 m_2}$$

$$m_1 = m_2 = 9ml \qquad k_1 = k_4 = kl + 3mg \qquad k_2 = k_3 = -kl$$

$$\omega^2 = \frac{kl + 3mg \pm kl}{9ml} \qquad \boxed{\omega_1 = \sqrt{\frac{g}{3l}}} \qquad \boxed{\omega_2 = \sqrt{\frac{2k}{9m} + \frac{g}{3l}}}$$

$$\frac{a_1}{a_2} = \frac{kl}{kl + 3mg - 9ml\omega^2} \qquad \boxed{\left[\frac{a_1}{a_2}\right]_1 = 1} \qquad \boxed{\left[\frac{a_1}{a_2}\right]_2 = -1}$$

Problem 4 (30 points)



Two masses, M_1 and M_2, slide on massless rollers in a bent track such that they cannot rotate with respect to the track. They are connected via an inextensible massless string which wraps without slip around a circular pulley of uniform density, mass m, and radius r. M_1 is connected to a fixed wall to the left by a spring k and a dashpot c. M_2 moves along a track rotated by a fixed angle θ_0 and moves under an applied external horizontal force $F(t)\,\hat{\mathbf{I}}$ and gravity $-g\,\hat{\mathbf{J}}$. For simplicity, assume that we only consider motion for which the string is under tension.

(a) (5 points) How many degrees of freedom n does this system have? Define a complete and independent set of scalar generalized coordinate(s) $(q_1, ..., q_n)$ for this system.

(b) (10 points) Write down the Lagrangian for this system.

(c) (10 points) Write down the generalized force(s) associated with each generalized coordinate for this system.

(d) (5 points) Find the equation(s) of motion for this system.

(a) (5 points) How many degrees of freedom n does this system have? Define a complete and independent set of scalar generalized coordinate(s) $(q_1, ..., q_n)$ for this system.

This problem is a one degree of freedom problem, thus $n = 1$. A natural variable that fully describe the motion of this system is a variable parameterizing the linear displacement of M_1 from the spring's un-stretched postion.

Define distance x as the distance M_1 has moved from the spring's un-stretched position.

(b) (10 points) Write down the Lagrangian for this system.

$$T = \frac{1}{2}M_1\dot{x}^2 + \frac{1}{2}M_2\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{\dot{x}}{r}\right)^2$$

$$V = \frac{1}{2}kx^2 - M_2 g x \sin\theta_0$$

$$\boxed{L = \frac{1}{2}\left(M_1 + M_2 + \frac{1}{2}m\right)\dot{x}^2 - \frac{1}{2}kx^2 + M_2 g x \sin\theta_0}$$

(c) (10 points) Write down the generalized force(s) associated with each generalized coordinate for this system.

$$\Xi_x = \sum_{i=1}^{N} \boldsymbol{F}_i^{nc} \cdot \frac{\partial}{\partial x} {}^{\mathrm{O}}\boldsymbol{r}_i = -c\dot{x} + F(t)\,\hat{\mathbf{I}} \cdot (\cos\theta_0\,\hat{\mathbf{I}} - \sin\theta_0\,\hat{\mathbf{J}}) = \boxed{-c\dot{x} + F(t)\cos\theta_0}$$

(d) (5 points) Find the equation(s) of motion for this system.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = \Xi_x$$

$$\left(M_1 + M_2 + \frac{1}{2}m\right)\ddot{x} - (-kx + M_2 g \sin\theta_0) = -c\dot{x} + F(t)\cos\theta_0$$

$$\boxed{\left(M_1 + M_2 + \frac{1}{2}m\right)\ddot{x} + c\dot{x} + kx = M_2 g \sin\theta_0 + F(t)\cos\theta_0}$$