2.003 Quiz #1 Review

1 Reference Frames

Reference frames are defined by an origin and right-handed set of unit vectors. We assume there exists some inertial reference frame, typically $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, which is fixed in time. Reference frames in general have six degrees of freedom with respect to other reference frames. They can translate with three degrees of freedom and can rotate with another three.

When we specify an intermediate frame $\hat{\mathrm{A}} = (A, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A)$, we must define the translation of its origin, ${}^{\mathrm{O}}\boldsymbol{r}_A(t)$, and the rotation of its unit vectors $\hat{\mathbf{i}}_A$, $\hat{\mathbf{j}}_A$, and $\hat{\mathbf{k}}_A$ as functions of $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, $\hat{\mathbf{K}}$, and the angles of the problem. Typically, it is sufficient to say that the intermediate frame is fixed to some rigid body and draw its unit vectors in relation to the ground.

We use reference frames to help us deal with rotation. As such, we typically only need to define one intermediate frame with respect each rotation in a problem. If there are two rotations, you will generally use three reference frames: one ground, and two intermediate.

2 Variables of Motion

Variables of motion are scalar distances or angles that change in time and define how the motion of the system is parameterized. The equations of motion of a system will be differential equations of the variables of motion. We will talk about defining a complete and independent set of generalized coordinates when we introduce the variational method of finding the equations of motion, but, for now, expect to be given what these variables must be.

3 Newton's Laws

In classical mechanics, we except without proof Newton's three laws of motion. They are:

(1) If no forces act on a particle, it will move in a straight line at constant or zero velocity.

(2) $\sum \boldsymbol{F}_m = \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{O}}\boldsymbol{p}_m)$ for mass m, inertial frame $\hat{\mathrm{O}}$, and ${}^{\mathrm{O}}\boldsymbol{p}_m = m{}^{\mathrm{O}}\boldsymbol{v}_m$.

(3) Reaction forces between particles are equal in magnitude and opposite in direction.

The first law allows us to define an inertial (or approximately inertial) reference frame. It is useful to note that the second law is a fundamental law, while the relationship between torques and angular momentum is derivative. Let us derive it.

Define $\boldsymbol{\tau}_m^B = {}^{\mathrm{B}}\boldsymbol{r}_m \times \boldsymbol{F}_m$ and ${}^{O}\boldsymbol{h}_m^B = {}^{\mathrm{B}}\boldsymbol{r}_m \times {}^{\mathrm{O}}\boldsymbol{p}_m$. Then:

$$
{}^{\mathrm{B}}\boldsymbol{r}_m \times \left(\sum \boldsymbol{F}_m = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{p}_m\right) \right) \quad \Longrightarrow \quad
\begin{aligned}
\sum \boldsymbol{\tau}_m^B &= {}^{\mathrm{B}}\boldsymbol{r}_m \times \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{p}_m\right) \\
&= \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{B}}\boldsymbol{r}_m \times {}^{\mathrm{O}}\boldsymbol{p}_m\right) - \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{B}}\boldsymbol{r}_m\right) \times {}^{\mathrm{O}}\boldsymbol{p}_m \\
&= \frac{{}^{\mathrm{O}}d}{dt}\left({}^{O}\boldsymbol{h}_m^B\right) - \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{O}}\boldsymbol{r}_m - {}^{\mathrm{O}}\boldsymbol{r}_B\right) \times {}^{\mathrm{O}}\boldsymbol{p}_m \\
&= \frac{{}^{\mathrm{O}}d}{dt}\left({}^{O}\boldsymbol{h}_m^B\right) - \left({}^{\mathrm{O}}\boldsymbol{v}_m - {}^{\mathrm{O}}\boldsymbol{v}_B\right) \times {}^{\mathrm{O}}\boldsymbol{p}_m
\end{aligned}
$$

Since ${}^{\mathrm{O}}\boldsymbol{v}_m$ is parallel to ${}^{\mathrm{O}}\boldsymbol{p}_m$, ${}^{\mathrm{O}}\boldsymbol{v}_m \times {}^{\mathrm{O}}\boldsymbol{p}_m = 0$, and:

$$
\sum \boldsymbol{\tau}_m^B = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{O}\boldsymbol{h}_m^B\right) + {}^{\mathrm{O}}\boldsymbol{v}_B \times {}^{\mathrm{O}}\boldsymbol{p}_m
$$

Note that the traditional notion that the sum of the torques is equal to the change in the angular momentum requires that the pivot B be stationary or that the velocity of the pivot and the linear momentum of the mass be parallel.

We have not learned how to apply Newton's second to distributed masses, but we have learned how to apply Newton's second law to both point masses *and* massless rigid bodies. It can be helpful to write down Newton's second law for every moving body in a problem, which may be necessary to find enough equations to solve for the unknowns.

4 Kinematics

We derived the following mathematical definition for taking a derivative of a vector defined in an intermediate frame:

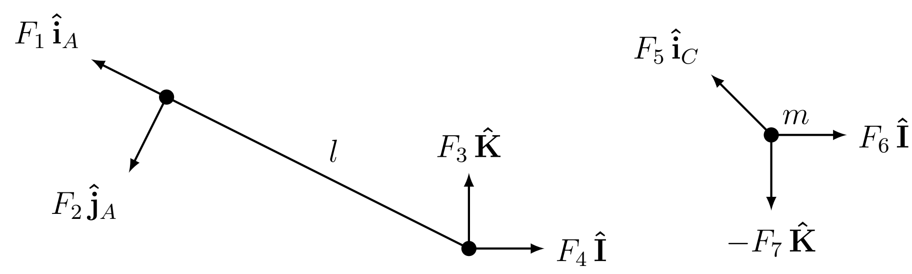
$$
\frac{{}^{\mathrm{O}}d}{dt}{}^{\mathrm{A}}\boldsymbol{r}_P = {}^{\mathrm{A}}\dot{\boldsymbol{r}}_P + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_P
$$

From this equation, we can derive velocities and accelerations in any frame by taking derivatives of position vectors. For one intermediate reference frame, the velocity and acceleration equations are:

$$
\begin{aligned}
{}^{\mathrm{O}}\boldsymbol{v}_P &= {}^{\mathrm{O}}\dot{\boldsymbol{r}}_A + {}^{\mathrm{A}}\dot{\boldsymbol{r}}_P + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_P \\
{}^{\mathrm{O}}\boldsymbol{a}_P &= {}^{\mathrm{O}}\ddot{\boldsymbol{r}}_A + {}^{\mathrm{A}}\ddot{\boldsymbol{r}}_P + 2\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\dot{\boldsymbol{r}}_P + {}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_P + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times \left({}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_P\right)
\end{aligned}
$$

The first two terms are translational accelerations, the third is the Coriolis acceleration, the fourth is the Euler acceleration, and the last is the centripetal acceleration.

5 Free Body Diagrams



When drawing free body diagrams, it is good practice to write both the magnitude and direction of forces in terms of the unit vectors of the your reference frames. This will make your statement of directions precise and independent of your ability to draw accurately, especially for drawings in three dimensions. Also remember that reaction forces often constrain more than one degree of freedom, requiring more than one scalar variable to define. Reaction forces are always equal in magnitude and opposite in direction under Newton's third law.

6 Equations of Motion

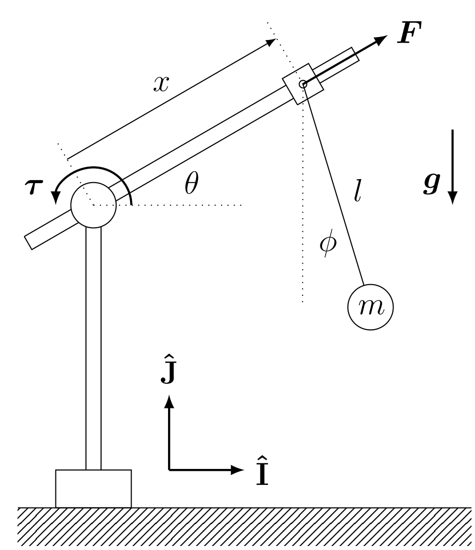
The equations of motion are a system of scalar differential equations in the variables of motion and their derivatives. In the direct method which we have learned, these equations come from Newton's second law. For any given problem, Newton's second law will contain a combination of known and unknown forces, torques, and variables of motion. If the movement is known, we can solve for forces and torques (Type 1 problem). If the forces and torques are known, we can solve for the equations of motion (Type 2 problem). If some mix of variables are known, we can solve equations for the remainder (Hybrid Type 3 problem).

7 Example: Crane

Consider a crane moving a payload of mass m in the presence of gravity. The crane has three degrees of freedom: the crane arm can rotate about a fixed pivot, a translating platform can move along the crane arm, and the mass swings from this platform. Assume the mass of the crane arm and connecting cable are negligible compared to the mass of the payload.

(a) Given that the crane moves with a specified $x(t)$, $\theta(t)$, and $\phi(t)$, what is the tension T on the cable?

(b) Given that a known torque $\boldsymbol{\tau} = \tau\,\hat{\mathbf{K}}$ is applied to the crane arm, and the crane arm exerts a known force $\boldsymbol{F} = F\,\hat{\mathbf{i}}_A$ on the platform, derive the equations of motion.



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Solution

For both questions, we must apply Newton's second law to find equations in our knowns to find our unknowns. Let us find these equations, and then see how they relate to the questions.

First, define reference frames. Define ground frame $\hat{\mathrm{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$ with O located at the first pivot point of the crane which is fixed in time. Also define intermediate frames $\hat{\mathrm{A}} = (A \equiv O, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A \equiv \hat{\mathbf{K}})$ rotating with the crane arm and frame $\hat{\mathrm{B}} = (B, \hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B \equiv \hat{\mathbf{K}})$ rotating with the cable with origin B at the cable's point of contact with the crane arm. For convenience, let $\hat{\mathbf{i}}_A$ point from O to B, and $\hat{\mathbf{i}}_B$ point from B to m. Note that the variables of motion are $x(t)$, $\theta(t)$, and $\phi(t)$.

Now let us write down Newton's second law as it applies to each rigid body in the problem: the arm a, the cable c, and the mass m. The mass is a point mass, which should lead to three independent equations, and the arm and the cable are rigid bodies, so these should lead to six independent equations each. Note that the arm and cable rigid bodies are massless, so their linear and angular momentums must be zero.

$$\sum \boldsymbol{F}_m = \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{O}}\boldsymbol{p}_m) = m\,{}^{\mathrm{O}}\boldsymbol{a}_m$$

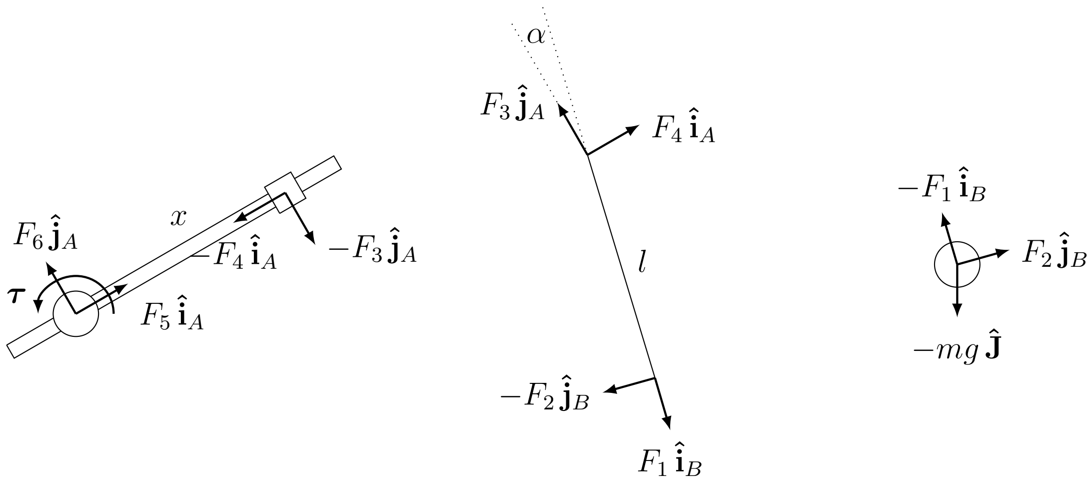
$$\sum \boldsymbol{F}_c = \frac{{}^{\mathrm{O}}d}{dt}(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{p}_c}) = \mathbf{0} \qquad \sum \boldsymbol{\tau}_c^B = \frac{{}^{\mathrm{O}}d}{dt}\left(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{h}_c^B}\right) + {}^{\mathrm{O}}\boldsymbol{v}_B \times \cancelto{0}{{}^{\mathrm{O}}\boldsymbol{p}_c} = \mathbf{0}$$

$$\sum \boldsymbol{F}_a = \frac{{}^{\mathrm{O}}d}{dt}(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{p}_a}) = \mathbf{0} \qquad \sum \boldsymbol{\tau}_a^O = \frac{{}^{\mathrm{O}}d}{dt}\left(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{h}_a^O}\right) + \cancelto{0}{{}^{\mathrm{O}}\boldsymbol{v}_O} \times \cancelto{0}{{}^{\mathrm{O}}\boldsymbol{p}_a} = \mathbf{0}$$

To solve these equations, we must draw free body diagrams to find the left side of the equations, and kinematics to find the right side of the equations. Let us start with kinematics, as there is only one non-zero right hand side.

$$\begin{aligned}
{}^{\mathrm{O}}\boldsymbol{a}_m &= \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{O}}\boldsymbol{r}_m)\right) = \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{{}^{\mathrm{O}}d}{dt}(\cancelto{0}{{}^{\mathrm{O}}\boldsymbol{r}_A} + {}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{B}}\boldsymbol{r}_m)\right) \\
&= \frac{{}^{\mathrm{O}}d}{dt}\left({}^{\mathrm{A}}\dot{\boldsymbol{r}}_B + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + \cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_m} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_m\right) \\
&= {}^{\mathrm{A}}\ddot{\boldsymbol{r}}_B + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\dot{\boldsymbol{r}}_B + {}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times ({}^{\mathrm{A}}\dot{\boldsymbol{r}}_B + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} \times {}^{\mathrm{A}}\boldsymbol{r}_B) + {}^{\mathrm{O}}\dot{\boldsymbol{\omega}}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_m + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times (\cancelto{0}{{}^{\mathrm{B}}\dot{\boldsymbol{r}}_m} + {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \times {}^{\mathrm{B}}\boldsymbol{r}_m) \\
&= (\ddot{x}\,\hat{\mathbf{i}}_A) + (\dot{\theta}\dot{x}\,\hat{\mathbf{j}}_A) + (\ddot{\theta}x\,\hat{\mathbf{j}}_A) + (\dot{\theta}\dot{x}\,\hat{\mathbf{j}}_A) - (\dot{\theta}^2 x\,\hat{\mathbf{i}}_A) + (\ddot{\phi}l\,\hat{\mathbf{j}}_B) - (\dot{\phi}^2 l\,\hat{\mathbf{i}}_B) \\
&= \ddot{x}\,\hat{\mathbf{i}}_A + 2\dot{\theta}\dot{x}\,\hat{\mathbf{j}}_A + \ddot{\theta}x\,\hat{\mathbf{j}}_A - \dot{\theta}^2 x\,\hat{\mathbf{i}}_A + \ddot{\phi}l\,\hat{\mathbf{j}}_B - \dot{\phi}^2 l\,\hat{\mathbf{i}}_B
\end{aligned}$$

Looking at the terms, we have a translational acceleration, a Coriolis term from the movement of B in the $\hat{\mathrm{A}}$ reference frame, and an Euler and centripetal acceleration for each rotation. Kinematics is done. Now we need to draw force body diagrams for the left sides of our equations. It will be useful to know the angle α between $\hat{\mathbf{j}}_A$ and $-\hat{\mathbf{i}}_B$, such that $-\hat{\mathbf{i}}_B \times \hat{\mathbf{j}}_A = \sin\alpha\,\hat{\mathbf{K}}$. A little geometry reveals $\alpha = \theta - \phi$.



Note that $F_4\,\hat{\mathbf{i}}_A = \boldsymbol{F} = F\,\hat{\mathbf{i}}_A$ and $\boldsymbol{\tau} = \tau\,\hat{\mathbf{K}}$.

$$\sum \boldsymbol{F}_a = F_5\,\hat{\mathbf{i}}_A + F_6\,\hat{\mathbf{j}}_A - F\,\hat{\mathbf{i}}_A - F_3\,\hat{\mathbf{j}}_A = \mathbf{0} \qquad F_5 = F \qquad F_6 = F_3$$

$$\sum \boldsymbol{\tau}_a^O = -xF_3\hat{\mathbf{K}} + \tau\,\hat{\mathbf{K}} = \mathbf{0} \qquad F_3 = \frac{\tau}{x}$$

$$\sum \boldsymbol{\tau}_c^B = -lF_2\hat{\mathbf{K}} = \mathbf{0} \qquad F_2 = 0$$

$$\sum \boldsymbol{F}_c = \boldsymbol{F} + F_3\,\hat{\mathbf{j}}_A + F_1\,\hat{\mathbf{i}}_B - \cancelto{0}{F_2}\,\hat{\mathbf{j}}_B = \mathbf{0} \qquad F_1\cos(\theta-\phi) = \frac{\tau}{x} \qquad F_1\sin(\theta-\phi) = F$$

$$\sum \boldsymbol{F}_m = -F_1\,\hat{\mathbf{i}}_B + \cancelto{0}{F_2}\,\hat{\mathbf{j}}_B - mg\,\hat{\mathbf{J}} = m\,{}^{\mathrm{O}}\boldsymbol{a}_m \qquad -F_1\,\hat{\mathbf{i}}_B - mg\,\hat{\mathbf{J}} = m\,{}^{\mathrm{O}}\boldsymbol{a}_m$$

Note that F_1 is the tension on the cable, with $F_1 = T = \sqrt{F^2 + \frac{\tau^2}{x^2}}$

Thus, for question (a):

$$T = -m({}^{\mathrm{O}}\boldsymbol{a}_m + g\,\hat{\mathbf{J}}) \cdot \hat{\mathbf{i}}_B$$
$$T = -m[\ddot{x}\,\hat{\mathbf{i}}_A + 2\dot{\theta}\dot{x}\,\hat{\mathbf{j}}_A + \ddot{\theta}x\,\hat{\mathbf{j}}_A - \dot{\theta}^2x\,\hat{\mathbf{i}}_A + \ddot{\phi}l\,\hat{\mathbf{j}}_B - \dot{\phi}^2l\,\hat{\mathbf{i}}_B + g(\sin\phi\hat{\mathbf{j}}_B - \cos\phi\hat{\mathbf{i}}_B)] \cdot \hat{\mathbf{i}}_B$$

$$\boxed{T = m[(\ddot{x} - \dot{\theta}^2x)\sin(\theta-\phi) + (2\dot{\theta}\dot{x} + \ddot{\theta}x)\cos(\theta-\phi) + \dot{\phi}^2l + g\cos\phi]}$$

And for question (b):

$$\sqrt{F^2 + \frac{\tau^2}{x^2}}\,\hat{\mathbf{i}}_B = m({}^{\mathrm{O}}\boldsymbol{a}_m + g\,\hat{\mathbf{J}}) \qquad \boxed{\sqrt{F^2 + \frac{\tau^2}{x^2}}\sin(\theta-\phi) = F}$$

$$\sqrt{F^2 + \frac{\tau^2}{x^2}}\,\hat{\mathbf{i}}_B = m[\ddot{x}\,\hat{\mathbf{i}}_A + 2\dot{\theta}\dot{x}\,\hat{\mathbf{j}}_A + \ddot{\theta}x\,\hat{\mathbf{j}}_A - \dot{\theta}^2x\,\hat{\mathbf{i}}_A + \ddot{\phi}l\,\hat{\mathbf{j}}_B - \dot{\phi}^2l\,\hat{\mathbf{i}}_B + g(\sin\phi\hat{\mathbf{j}}_B - \cos\phi\hat{\mathbf{i}}_B)]$$

$$\boxed{\sqrt{F^2 + \frac{\tau^2}{x^2}} = m[(\ddot{x} - \dot{\theta}^2x)\sin(\theta-\phi) + (2\dot{\theta}\dot{x} + \ddot{\theta}x)\cos(\theta-\phi) + \dot{\phi}^2l + g\cos\phi]}$$

$$\boxed{0 = m[-(\ddot{x} - \dot{\theta}^2x)\cos(\theta-\phi) - (2\dot{\theta}\dot{x} + \ddot{\theta}x)\sin(\theta-\phi) + \ddot{\phi}l - g\sin\phi]}$$

2.003 Quiz #2 Review

1 Basic Concepts

Constraints

In general, connections can constrain possible movement between bodies. A rigid connection means that an arbitrary reaction force and torque can be sustained in any direction, leading to six scalar reaction unknowns and zero degrees of freedom. For planar motion, this becomes three scalar unknowns and zero degrees of freedom. A pinned connection means that in general an arbitrary reaction force can be sustained in any direction, but no torque, leading to three scaler reaction unknowns and three rotational degrees of freedom. For planar motion, this becomes two scalar unknowns and one translational degree of freedom. A surface connection means that the motion must follow a surface or plane which constrains the object's rotation, and movement in the normal direction, leading to four scalar reaction unknowns and two translational degrees of freedom. For planar motion, this becomes two scalar unknowns and one translational degree of freedom.

Constitutive Relations

Force from gravity is given by $\boldsymbol{F}_g = m\boldsymbol{g}$. Force in a spring with spring constant k obeys Hooke's Law $\boldsymbol{F}_S = -k\boldsymbol{x}$ where $\boldsymbol{x}$ is the displacement from the spring's un-stretched length. Force related to a dashpot or viscous drag with viscous damping coefficient c can be assumed to obey the following relationship, $\boldsymbol{F}_D = -c\boldsymbol{v}$ where $\boldsymbol{v}$ is either the velocity of the object experiencing drag or the difference in velocity between the two ends of a dashpot. Force from kinetic friction is usually given by a $\boldsymbol{F}_f = -\mu N \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ where μ is a constant related to the contact surface pair, N is the normal force between the surfaces, and $-\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is the direction opposite the object's movement. Note that for kinetic friction to act, surfaces must slide relative to each other. Rolling is no-slip, thus kinetic friction is not applicable.

Tensor Review

Define frame $\hat{\text{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$ and frame $\hat{\text{A}} = (O, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \equiv \hat{\mathbf{K}})$ with $\hat{\mathbf{i}} = \cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}}$. Define a tensor $\hat{\mathbf{i}} \otimes \hat{\mathbf{i}}$. We can write this tensor as:

$$\hat{\mathbf{i}} \otimes \hat{\mathbf{i}} = {}^{A}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (\cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}}) \otimes (\cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}}) = {}^{O}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also, $(\boldsymbol{A} \otimes \boldsymbol{B}) \cdot \boldsymbol{C} = (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A}$, thus $\quad (\hat{\mathbf{I}} \otimes \hat{\mathbf{I}}) \cdot \hat{\mathbf{I}} = \hat{\mathbf{I}} \quad (\hat{\mathbf{I}} \otimes \hat{\mathbf{I}}) \cdot \hat{\mathbf{J}} = \mathbf{0} \quad (\hat{\mathbf{I}} \otimes \hat{\mathbf{I}}) \cdot \hat{\mathbf{i}} = \cos\theta\,\hat{\mathbf{I}}$

2 Direct Method

Newton's Second Law for Rigid Bodies

$$\sum \boldsymbol{F}_M = M\,{}^{\mathrm{O}}\boldsymbol{a}_C \quad \text{for} \quad {}^{\mathrm{O}}\boldsymbol{r}_C = \frac{1}{M}\iiint\limits_V \rho({}^{\mathrm{O}}\boldsymbol{r}_V)\,{}^{\mathrm{O}}\boldsymbol{r}_V\,dV$$

This must be applied to the center of mass C in an inertial frame $\hat{\mathrm{O}}$. M refers to the total mass of the rigid body and $\rho({}^{\mathrm{O}}\boldsymbol{r}_V)$ represents the density of the body as a function of position. If the locations of the center of mass of the individual parts of a body are known, the location of the center of mass of the whole body is just the weighted average of the part centers.

Euler's Equations for Rigid Bodies

$$\sum \boldsymbol{\tau}_M^B = \frac{{}^{\mathrm{O}}d}{dt}\left({}^{O}\boldsymbol{H}_M^B\right) + \underline{M\,{}^{\mathrm{O}}\boldsymbol{v}_B \times {}^{\mathrm{O}}\boldsymbol{v}_C} \quad \text{for} \quad {}^{O}\boldsymbol{H}_M^B = \bar{\mathbf{I}}_M^B\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} + \underline{M\,{}^{\mathrm{B}}\boldsymbol{r}_C \times {}^{\mathrm{O}}\boldsymbol{v}_B}$$

This must be applied in an inertial frame $\hat{\mathrm{O}}$, but the point of rotation B is arbitrary. M refers to the total mass of the rigid body while ${}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}}$ refers to a frame $\hat{\mathrm{B}}$ fixed to the rigid body. The underlined terms go to zero when either $B \equiv C$ <u>or</u> ${}^{\mathrm{O}}\boldsymbol{v}_B = \mathbf{0}$.

3 Moment of Inertia Tensor

Definition for ${}^{\mathrm{B}}\boldsymbol{r}_{dV} = x\,\hat{\mathbf{i}}_A + y\,\hat{\mathbf{j}}_A + z\,\hat{\mathbf{k}}_A$:

$$\bar{\mathbf{I}}_M^B = \iiint\limits_V \rho({}^{\mathrm{B}}\boldsymbol{r}_{dV})\left[({}^{\mathrm{B}}\boldsymbol{r}_{dV}\cdot{}^{\mathrm{B}}\boldsymbol{r}_{dV})\mathbb{1}_3 - ({}^{\mathrm{B}}\boldsymbol{r}_{dV}\otimes{}^{\mathrm{B}}\boldsymbol{r}_{dV})\right]dV$$

$$= \iiint\limits_V \rho(x,y,z)\;{}^{A}\!\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & z^2+x^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} dx\,dy\,dz$$

While the inertia tensor does not depend on a reference frame, we must specify a reference frame (here frame $\hat{\mathrm{A}}$) to write it in matrix form. The point of rotation B is arbitrary.

Parallel Axis Theorem

Parallel Axis Theorem for ${}^{\mathrm{B}}\boldsymbol{r}_C = a\,\hat{\mathbf{i}}_A + b\,\hat{\mathbf{j}}_A + c\,\hat{\mathbf{k}}_A$:

$$\bar{\mathbf{I}}_M^B = \bar{\mathbf{I}}_M^C + M\left[({}^{\mathrm{B}}\boldsymbol{r}_C\cdot{}^{\mathrm{B}}\boldsymbol{r}_C)\mathbb{1}_3 - ({}^{\mathrm{B}}\boldsymbol{r}_C\otimes{}^{\mathrm{B}}\boldsymbol{r}_C)\right] = \bar{\mathbf{I}}_M^C + M\;{}^{A}\!\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ba & c^2+a^2 & -bc \\ -ca & -cb & a^2+b^2 \end{bmatrix}$$

The parallel axis theorem only applies when shifting from the rigid body's center of mass C. Also note that you can use offset vector ${}^{\mathrm{B}}\boldsymbol{r}_C$ or ${}^{\mathrm{C}}\boldsymbol{r}_B = -{}^{\mathrm{B}}\boldsymbol{r}_C$ as the equations will be the same either way due to squared nature of the moment of inertia.

Principal Axes

In general, a matrix $\overline{\boldsymbol{A}}$ times a vector $\boldsymbol{b}$ yields a vector $\overline{\boldsymbol{A}}\boldsymbol{b} = \boldsymbol{c}$ pointing in a different direction than the original vector $\boldsymbol{b}$. However, an eigenvector $\boldsymbol{x}$ of $\overline{\boldsymbol{A}}$ has the property such that the product will point in the same direction as the original vector $\overline{\boldsymbol{A}}\boldsymbol{x} = \lambda\boldsymbol{x}$. Here λ is the eigenvalue corresponding to the eigenvector $\boldsymbol{x}$ of the matrix $\overline{\boldsymbol{A}}$.

Define the principal frame of a body about B to be $\hat{\mathrm{B}} = (B, \hat{\mathbf{i}}_B, \hat{\mathbf{j}}_B, \hat{\mathbf{k}}_B)$. The principal axes $\hat{\mathbf{i}}_B$, $\hat{\mathbf{j}}_B$, and $\hat{\mathbf{k}}_B$ are eigenvectors of a moment of inertia tensor $\bar{\mathbf{I}}\,^B_M$ such that $\bar{\mathbf{I}}\,^B_M\,\hat{\mathbf{i}}_B = I^B_{xx}\,\hat{\mathbf{i}}_B$, $\bar{\mathbf{I}}\,^B_M\,\hat{\mathbf{j}}_B = I^B_{yy}\,\hat{\mathbf{j}}_B$, and $\bar{\mathbf{I}}\,^B_M\,\hat{\mathbf{k}}_B = I^B_{zz}\,\hat{\mathbf{k}}_B$. This is the same as saying that the inertia tensor written in terms of these directions yields a diagonal matrix. Here we call the eigenvalues I^B_{xx}, I^B_{yy}, and I^B_{zz} the principal moments. When you look up moments of inertia in a table, you are given the principal moments in the principal directions taken about the centroid of the body: I^C_{xx}, I^C_{yy}, and I^C_{zz}.

Rigid Body Rotations About Principal Axes

If a rigid body only undergoes rotation about a principal axis $\hat{\mathbf{K}}$ such that ${}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} = \omega\,\hat{\mathbf{K}}$, only the $\hat{\mathbf{K}} \otimes \hat{\mathbf{K}}$ component of inertia tensor will contribute to the product $\bar{\mathbf{I}}\,^B_m\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}}$ in Euler's Equations because $(\hat{\mathbf{I}} \otimes \hat{\mathbf{I}}) \cdot \hat{\mathbf{K}} = (\hat{\mathbf{J}} \otimes \hat{\mathbf{J}}) \cdot \hat{\mathbf{K}} = \mathbf{0}$. Thus, if rotations are about a principal axis $\hat{\mathbf{K}}$, you can write the following:

$$\bar{\mathbf{I}}\,^B_M\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} = I^B_{zz}\,\omega\,\hat{\mathbf{K}} \qquad \text{for} \qquad I^B_{zz} = \bar{\mathbf{I}}\,^B_M \cdot (\hat{\mathbf{K}} \otimes \hat{\mathbf{K}})$$

Also when rotation is about a principal axis $\hat{\mathbf{K}}$, for any r that offsets the body's center of mass C perpendicular to the principal axis to a point B such that $r = |{}^{\mathrm{B}}\boldsymbol{r}_C|$ for ${}^{\mathrm{B}}\boldsymbol{r}_C \cdot \hat{\mathbf{K}} = 0$, the parallel axis theorem reduces to the following.

$$I^B_{zz} = I^C_{zz} + mr^2$$

4 Variational Method

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \Xi_j$$

Here, j varies from 1 to n, where n is the number of degrees of freedom of the system, q_j refers to the jth generalized coordinate, Ξ_j refers to the jth generalized force, and the Lagrangian $\mathcal{L} = T - V$ is the difference between the kinetic energy T and conservative potential energy V of the system. Lagrange's Equations give a total of n equations of motion: one for each degree of freedom of the problem. Note that each generalized coordinate will always be either a scalar length or angle. Also note that Lagrange's equations in this form are restricted to holonomic systems, and can only be used to find equations of motion, not reaction forces, which must be found using the direct method.

Kinetic Energy

Kinetic energy is the energy associated with motion. Translational kinetic energy is the energy associated with the movement of the center of mass C of a rigid body m, while rotational kinetic energy is the energy associated with the rotation of the rigid body about C. As with the angular momentum, you must take these energies with respect to the movement of and around either the center of mass C or around a stationary point B with respect to an inertial frame Ô. Let ${}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}}$ represent the angular velocity of frame B̂ fixed to the body relative to the ground. Then for either $B \equiv C$ or ${}^{\mathrm{O}}\boldsymbol{v}_B = \mathbf{0}$:

$$T_{translational} = \frac{1}{2} m({}^{\mathrm{O}}\boldsymbol{v}_B \cdot {}^{\mathrm{O}}\boldsymbol{v}_B) \qquad T_{rotational} = \frac{1}{2} {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}} \cdot (\bar{\mathbf{I}}_m^{B} \, {}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{B}})$$

Potential Energy

Potential energy comes from forces that depend only on an object's position or orientation, and does not change in time or velocity. We will consider potential energy in two forms: gravitational potential energy and spring potential energy. We will use generalized forces to express all other types of forces. Gravitational potential energy is proportional to the change in position of the center of mass C of a body m in the direction of gravity. Spring potential energy is proportional to the magnitude squared of the displacement $\boldsymbol{x}$ from the spring's un-stretched length.

$$V_{gravitational} = -m\boldsymbol{g} \cdot {}^{\mathrm{O}}\boldsymbol{r}_C \qquad V_{spring} = \frac{1}{2} k\boldsymbol{x} \cdot \boldsymbol{x}$$

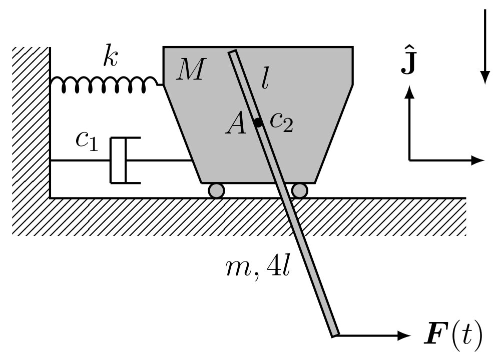
Generalized Forces

$$\Xi_j = \sum_{i=1}^{N} \boldsymbol{F}_i^{nc} \cdot \frac{\partial}{\partial q_j} {}^{\text{O}}\boldsymbol{r}_i$$

A generalized force Ξ_j is the variation of the work done on the system by nonconservative forces over a variational displacement of the jth generalized coordinate q_j. Again, j varies from 1 to n, where n is the number of degrees of freedom of the system, so there will be n generalized forces. Here i indexes all nonconservative forces $\boldsymbol{F}_i^{nc}$ acting on the system at locations ${}^{\text{O}}\boldsymbol{r}_i$. Here i varies from 1 to N, where N is the number of nonconservative forces acting on the system. Note that if q_j is a length, Ξ_j will have units of force, while if q_j is an angle, Ξ_j will have units of torque.

5 Example: Pendulum Cart

A cart of mass M is connected to a fixed wall to the left by a spring with spring constant k and a dashpot with damping coefficient c_1. The cart's wheels are of negligible mass and roll on frictionless bearings. A thin rod of mass m and length $4l$ is connected to the cart off-center via a pivot at point A that experiences viscous damping with damping coefficient c_2. A varying force $F(t)\,\hat{\mathbf{I}}$ is applied to the end of the bar. Find the equations of motion using both the direct and variational methods. What is the normal force acting on the cart?



Solution

Indirect Method Equations of Motion

Two degree of freedom problem. Let $q_1 = x$ be the distance A has traveled relative to the un-stretched location of the end of the spring O, and let $q_2 = \theta$ be the angle the rod forms with the vertical such that ${}^{\text{A}}\boldsymbol{r}_C = l(\sin\theta\,\hat{\mathbf{I}} - \cos\theta\,\hat{\mathbf{J}})$. Define ground frame $\hat{\text{O}} = (O, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, and translating, rotating frame $\hat{\text{A}} = (A, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A \equiv \hat{\mathbf{K}})$ such that $\hat{\mathbf{j}}_A = -\sin\theta\,\hat{\mathbf{I}} + \cos\theta\,\hat{\mathbf{J}}$. Define C to be the position of the center of mass of the rod given by ${}^{\text{O}}\boldsymbol{r}_C = x\,\hat{\mathbf{I}} - l\,\hat{\mathbf{j}}_A$, and center of mass body frame $\hat{\text{C}} = (C, \hat{\mathbf{i}}_A, \hat{\mathbf{j}}_A, \hat{\mathbf{k}}_A \equiv \hat{\mathbf{K}})$. Note that ${}^{\text{O}}\boldsymbol{\omega}_{\text{A}} = {}^{\text{O}}\boldsymbol{\omega}_{\text{C}} = \dot{\theta}\,\hat{\mathbf{K}}$.

$$T = \frac{1}{2}M({}^{\text{O}}\boldsymbol{v}_A \cdot {}^{\text{O}}\boldsymbol{v}_A) + \frac{1}{2}M({}^{\text{O}}\boldsymbol{v}_C \cdot {}^{\text{O}}\boldsymbol{v}_C) + \frac{1}{2}{}^{\text{O}}\boldsymbol{\omega}_{\text{C}} \cdot (\bar{\mathbf{I}}_m^{C}\,{}^{\text{O}}\boldsymbol{\omega}_{\text{C}}), \text{ from tables: } \bar{\mathbf{I}}_m^{C} = \frac{1}{12}m(4l)^2 \, {}^{C}\!\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$ {}^{\text{O}}\boldsymbol{v}_A = \dot{x}\,\hat{\mathbf{I}} \qquad {}^{\text{O}}\boldsymbol{v}_C = \frac{{}^{\text{O}}d}{dt}\left[x\,\hat{\mathbf{I}} - l(-\sin\theta\,\hat{\mathbf{I}} + \cos\theta\,\hat{\mathbf{J}})\right] = \dot{x}\,\hat{\mathbf{I}} + l\dot{\theta}(\cos\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}}) $$

$$ T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + l\dot{\theta}\cos\theta)^2 + l^2\dot{\theta}^2\sin^2\theta\right] + \frac{2}{3}\dot{\theta}^2 ml^2 \qquad V = \frac{1}{2}kx^2 + mg(-l\cos\theta) $$

$$ \mathcal{L} = T - V = \left[\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + \frac{2}{3}\dot{\theta}^2 ml^2\right] - \left(\frac{1}{2}kx^2 - mgl\cos\theta\right) $$

$$ \Xi_j = \sum_{i=1}^{N} \boldsymbol{F}_i^{nc} \cdot \frac{\partial}{\partial q_j}{}^{\text{O}}\boldsymbol{r}_i $$

Three nonconservative force acting: $\boldsymbol{F}(t)$ acting at D, ${}^{\text{O}}\boldsymbol{r}_D = x\,\hat{\mathbf{I}} - 3l\,\hat{\mathbf{j}}_A$

$\boldsymbol{F}_{D1} = -c_1\dot{x}\,\hat{\mathbf{I}}$ acting at A

A nonconservative torque equivalent to $\boldsymbol{\tau}_{D2} = -c_2\dot{\theta}\,\hat{\mathbf{K}}$

$$ \Xi_1 = F(t)\hat{\mathbf{I}} \cdot \frac{\partial}{\partial x}(x\,\hat{\mathbf{I}} - 3l\,\hat{\mathbf{j}}_A) + (-c_1\dot{x}\,\hat{\mathbf{I}}) \cdot \frac{\partial}{\partial x}(x\,\hat{\mathbf{I}}) = F(t) - c_1\dot{x} $$

$$ \Xi_2 = F(t)\hat{\mathbf{I}} \cdot \frac{\partial}{\partial \theta}(x\,\hat{\mathbf{I}} - 3l\,\hat{\mathbf{j}}_A) + (-c_1\dot{x}\,\hat{\mathbf{I}}) \cdot \frac{\partial}{\partial \theta}(x\,\hat{\mathbf{I}}) - c_2\dot{\theta} = 3F(t)l\cos\theta - c_2\dot{\theta} $$

$$ \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{\partial x} = \Xi_1 \qquad \frac{d}{dt}\Big(M\dot{x} + m(\dot{x} + l\dot{\theta}\cos\theta)\Big) - (-kx) = F(t) - c_1\dot{x} $$

$$ \boxed{(M+m)\ddot{x} + c_1\dot{x} + kx + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = F(t)} $$

$$ \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial \theta} = \Xi_2 \qquad \frac{d}{dt}\left(m\dot{x}l\cos\theta + \frac{7}{3}m\dot{\theta}l^2\right) - (-m\dot{x}l\dot{\theta}\sin\theta - mgl\sin\theta) = 3F(t)l\cos\theta - c_2\dot{\theta} $$

$$ \boxed{m\ddot{x}l\cos\theta + \frac{7}{3}m\ddot{\theta}l^2 + c_2\dot{\theta} + mgl\sin\theta = 3F(t)l\cos\theta} $$

Direct Method Equations of Motion

Cart M

Newton's second law $\quad \sum \boldsymbol{F}_M = M{}^{\text{O}}\boldsymbol{a}_A$

Note that the cart does not rotate, so Euler's equation does not help with equations of motion. Let $F_N\,\hat{\mathbf{J}}$ be the normal reaction forces between the ground and the cart, and let $F_1\,\hat{\mathbf{I}}$ and $F_2\,\hat{\mathbf{J}}$ be the reaction forces between the cart and the bar.

$$ \sum \boldsymbol{F}_M = -Mg\,\hat{\mathbf{J}} + F_N\,\hat{\mathbf{J}} + F_2\,\hat{\mathbf{J}} + F_1\,\hat{\mathbf{I}} - kx\,\hat{\mathbf{I}} - c\dot{x}\,\hat{\mathbf{I}} \qquad {}^{\text{O}}\boldsymbol{a}_A = \ddot{x}\,\hat{\mathbf{I}} $$

$$ -Mg\,\hat{\mathbf{J}} + F_N\,\hat{\mathbf{J}} + F_2\,\hat{\mathbf{J}} + F_1\,\hat{\mathbf{I}} - kx\,\hat{\mathbf{I}} - c\dot{x}\,\hat{\mathbf{I}} = M\ddot{x}\,\hat{\mathbf{I}} $$

$$ \underline{F_N + F_2 - Mg = 0} \qquad \underline{F_1 - kx - c\dot{x} = M\ddot{x}} $$

Rod m

Newton's second law $\sum \boldsymbol{F}_m = m{}^{\mathrm{O}}\boldsymbol{a}_C$

Euler's equations about point A $\sum \boldsymbol{\tau}_m^A = \frac{{}^{\mathrm{O}}d}{dt}\left(\bar{\mathbf{I}}_m^A\,{}^{\mathrm{O}}\boldsymbol{\omega}_{\mathrm{A}} + m\,{}^{\mathrm{A}}\boldsymbol{r}_C \times {}^{\mathrm{O}}\boldsymbol{v}_A\right) + m\,{}^{\mathrm{O}}\boldsymbol{v}_A \times {}^{\mathrm{O}}\boldsymbol{v}_C$

$$\sum \boldsymbol{F}_m = -mg\,\hat{\mathbf{J}} - F_2\,\hat{\mathbf{J}} - F_1\,\hat{\mathbf{I}} + F(t)\,\hat{\mathbf{I}}$$

We choose to take Euler's Equations about point A to avoid introducing reaction forces at the pivot around the center of mass. Note that the extra terms do not go to zero.

$${}^{\mathrm{O}}\boldsymbol{a}_C = \frac{{}^{\mathrm{O}}d}{dt}\left[\frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{O}}\boldsymbol{r}_A + {}^{\mathrm{A}}\boldsymbol{r}_C)\right] = \frac{{}^{\mathrm{O}}d}{dt}\left[\dot{x}\,\hat{\mathbf{I}} + \dot{\theta}l(\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}})\right] = (\ddot{x}+\ddot{\theta}l\cos\theta-\dot{\theta}^2 l\sin\theta)\hat{\mathbf{I}}+(\ddot{\theta}l\sin\theta+\dot{\theta}^2 l\cos\theta)\hat{\mathbf{J}}$$

$$\underline{F(t) - F_1 = m\ddot{x} + m\ddot{\theta}l\cos\theta - m\dot{\theta}^2 l\sin\theta} \qquad \underline{-mg - F_2 = m\ddot{\theta}l\sin\theta + m\dot{\theta}^2 l\cos\theta}$$

Combining above underlined equations yields one equation of motion.

$$\boxed{(M+m)\ddot{x} + c_1\dot{x} + kx + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = F(t)}$$

$$\sum \boldsymbol{\tau}_m^A = 3F(t)l\cos\theta\,\hat{\mathbf{K}} - mgl\sin\theta\,\hat{\mathbf{K}} - c_2\dot{\theta}\,\hat{\mathbf{K}}$$

$$\bar{\mathbf{I}}_m^A = \bar{\mathbf{I}}_m^C + m\,{}^{A}\!\begin{bmatrix} l^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & l^2 \end{bmatrix} = \frac{7}{3}ml^2\,{}^{A}\!\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{\mathrm{A}}\boldsymbol{r}_C \times {}^{\mathrm{O}}\boldsymbol{v}_A = (-l\,\hat{\mathbf{j}}_A) \times (\dot{x}\,\hat{\mathbf{I}}) = -\dot{x}l\cos\theta\hat{\mathbf{K}} \qquad {}^{\mathrm{O}}\boldsymbol{v}_A \times {}^{\mathrm{O}}\boldsymbol{v}_C = (\dot{x}\,\hat{\mathbf{I}}) \times (\dot{x}\,\hat{\mathbf{I}} + \dot{\theta}l(\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}})) = \dot{x}l\dot{\theta}\sin\theta\,\hat{\mathbf{K}}$$

$$3F(t)l\cos\theta\,\hat{\mathbf{K}} - mgl\sin\theta\,\hat{\mathbf{K}} - c_2\dot{\theta}\,\hat{\mathbf{K}} = \frac{{}^{\mathrm{O}}d}{dt}\left(\frac{7}{3}ml^2\dot{\theta}\,\hat{\mathbf{K}} - m\dot{x}l\cos\theta\hat{\mathbf{K}}\right) + m\dot{x}l\dot{\theta}\sin\theta\,\hat{\mathbf{K}}$$

$$\boxed{m\ddot{x}l\cos\theta + \frac{7}{3}m\ddot{\theta}l^2 + c_2\dot{\theta} + mgl\sin\theta = 3F(t)l\cos\theta}$$

Normal Force

Solving for F_N in the above equations yields:

$$\boxed{F_N = (M+m)g - m\ddot{\theta}l\sin\theta - m\dot{\theta}^2 l\cos\theta}$$

2.003 Final Exam Review

1 Finding Equations of Motion

The first two thirds of 2.003 focused on finding the equations of motion of mechanical systems. The direct method utilized Newton's second law and Euler's equation to find the equations of motion and the reaction forces in the system. The indirect method utilized Lagrange's equations and virtual work to find the equations of motion directly. Both methods require the use of kinematics to find velocities and accelerations. Please refer to the first two quiz reviews for more information concerning kinematics and finding equations of motion.

2 Equilibria, Linearization, and Stability

Most non-linear differential equations do not have analytical solutions. However, we would still like to be able to analyze non-linear systems, especially for small vibrations about stable equilibria. If we assume that displacements and velocities around the equilibrium points are small, we can linearize the equations of motion using Taylor series expansions.

2.1 Equilibrium points

Equilibrium requires that the system be at rest. By setting all velocities and accelerations in our equation of motion to zero, we find an equation for which equilibrium positions must be solutions. For example, if our equation of motion has the form:

$$f(x)\ddot{x} + g(x, \dot{x})\dot{x} + h(x) = 0$$

Equilibrium will be satisfied when $\dot{x} = \ddot{x} = 0$, and solutions to the equation $h(x) = 0$ will yield the m equilibrium points $(x_{eq}^{(1)}, x_{eq}^{(2)}, \ldots, x_{eq}^{(m)})$ of the non-linear system.

2.2 Linearization

In order to linearize an equation of motion about a point, we use Taylor series to approximate the equation to the first order, only keeping terms that are first order in a single variable of motion. While linearization derives from the Taylor series, it is sometimes useful and easier to linearize parts of the equation first, and then linearize the whole. For example, if our equation of motion has the form:

$$f(x)\ddot{x} + g(x,\dot{x})\dot{x} + h(x) = 0$$

Let's linearize each coefficient function about an equilibrium point. Let $x = x_{eq} + \epsilon$, $\dot{x} = \dot{\epsilon}$, $\ddot{x} = \ddot{\epsilon}$. Then:

$$f(x)\ddot{x} = f(x_{eq}+\epsilon)\ddot{\epsilon} = \left(f(x_{eq}) + \epsilon \left[\frac{\partial}{\partial x} f(x)\right]_{x=x_{eq}} + \dots \right)\ddot{\epsilon} \approx \underline{f(x_{eq})\ddot{\epsilon}}$$

For linear approximation, $\epsilon\ddot{\epsilon}$ and higher order terms are negligibly small compared to $\ddot{\epsilon}$.

$$g(x,\dot{x})\dot{x} = g(x_{eq}+\epsilon,\dot{\epsilon})\dot{\epsilon} = \left(g(x_{eq},0) + \epsilon \left[\frac{\partial}{\partial x} g(x,\dot{x})\right]_{\substack{x=x_{eq}\\ \dot{x}=0}} + \dot{\epsilon}\left[\frac{\partial}{\partial \dot{x}} g(x,\dot{x})\right]_{\substack{x=x_{eq}\\ \dot{x}=0}} + \dots \right)\dot{\epsilon} \approx \underline{g(x_{eq},0)\dot{\epsilon}}$$

For linear approximation, $\epsilon\dot{\epsilon}$, $\dot{\epsilon}^2$, and higher order terms are negligibly small compared to $\dot{\epsilon}$.

$$h(x_{eq}+\epsilon) = \left(h(x_{eq}) + \epsilon\left[\frac{\partial}{\partial x} h(x)\right]_{x=x_{eq}} + \dots \right) \approx \cancelto{0}{h(x_{eq})} + \underline{\epsilon\left[\frac{\partial}{\partial x} h(x)\right]_{x=x_{eq}}}$$

For linear approximation, ϵ^2 and higher order terms are negligibly small compared to ϵ. Note that by definition of an equilibrium point, $h(x_{eq}) = 0$.

Thus, for any single degree of freedom system that can be described in the form $f(x)\ddot{x}+g(x,\dot{x})\dot{x}+h(x) = 0$, the first order linear approximation about an equilibrium point x_{eq} can be written as:

$$\boxed{f(x_{eq})\ddot{\epsilon} + g(x_{eq},0)\dot{\epsilon} + \epsilon\left[\frac{\partial}{\partial x} h(x)\right]_{x=x_{eq}} = 0}$$

Note that $f(x_{eq})$, $g(x_{eq},0)$, and $\left[\frac{\partial}{\partial x}h(x)\right]_{x=x_{eq}} = 0$ are all <u>constants</u>. In a lumped parameter model, these would correspond to the m, c, and k coefficients respectively.

2.3 Stability

If a system is perturbed from an equilibrium position by a small amount and has a tendency to move back toward the equilibrium position, we call this system stable. If the system has a tendency to move away from the equilibrium position, we call this system unstable. We will see in the following section that second-order linear systems have exponential solutions. Stable systems correspond to exponential solutions that have a negative real part. It can be shown that second-order linear differential equations of the form $m\ddot{x} + c\dot{x} + kx = 0$ are stable if and only if all coefficients m, c, k have the same sign.

3 Solving Stable Second-order Linear Differential Equations

Stable second-order linear differential equations of motions have analytical solutions which are studied in 18.03. We call these systems lumped-parameter-systems because we can write them in the form $m\ddot{x} + c\dot{x} + kx = F(t)$. Solutions to these equations can be written as a sum of a homogeneous and particular solution $x(t) = x_h(t) + x_p(t)$. For convenience, let us define a natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ and a non-dimensional damping parameter $\zeta = \frac{c}{2\sqrt{mk}}$ such that our equation of motion can be written as $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t)/m$.

3.1 Homogeneous Solutions (Transient Response)

The homogeneous solution $x_h(t)$ solves the equation $\ddot{x}_h + 2\zeta\omega_n\dot{x}_h + \omega_n^2 x_h = 0$. Solutions to this equation are of the form $x_h(t) = Ae^{st}$. Plugging in yields the characteristic equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, which has solutions:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Overdamped System: $\zeta > 1$

When $\zeta > 1$, $\sqrt{\zeta^2 - 1}$ will be real, and both solutions of s will be real. The homogeneous solution will have the form:

$$\boxed{x_h(t) = Ae^{s_1 t} + Be^{s_2 t} \qquad \text{for} \qquad s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \qquad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}$$

Critically Damped System: $\zeta = 1$

When $\zeta = 1$, $\sqrt{\zeta^2 - 1}$ will be zero, and both solutions of s will be the same. The homogeneous solution will have the form:

$$\boxed{x_h(t) = (A + Bt)e^{-\omega_n t}}$$

Underdamped System: $0 < \zeta < 1$

When $0 < \zeta < 1$, $\sqrt{\zeta^2 - 1}$ will be imaginary, and both solutions of s will be imaginary. The homogeneous solution will have the form:

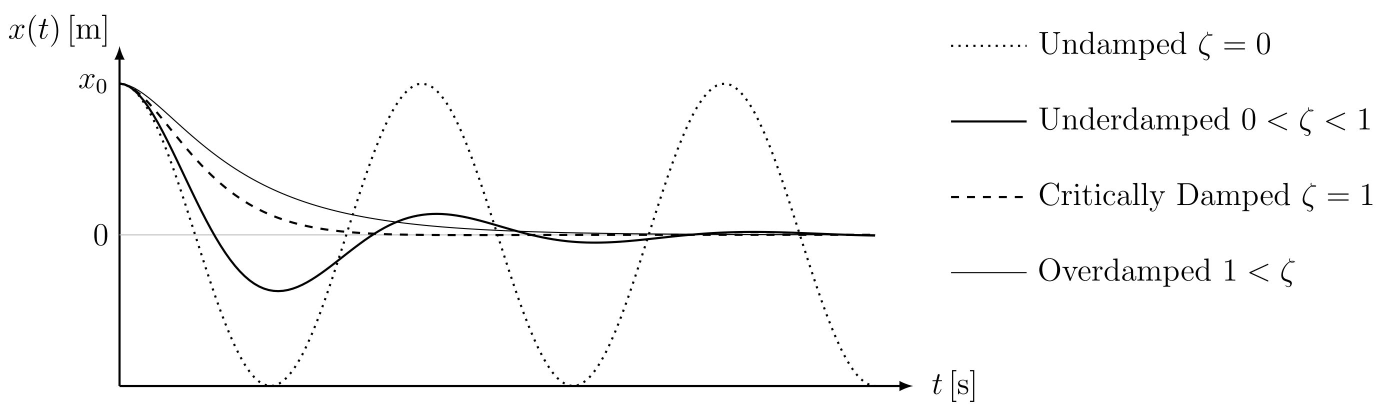
$$\boxed{x_h(t) = e^{-\zeta\omega_n t}[A\sin(\omega_d t) + B\cos(\omega_d t)] \qquad \text{for} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}}$$

Undamped System: $\zeta = 0$

When $\zeta = 0$, $\sqrt{\zeta^2 - 1} = i$. The homogeneous solution will have the form:

$$\boxed{x_h(t) = A\sin(\omega_n t) + B\cos(\omega_n t)}$$

Each of these homogeneous solutions represent a family of solutions determined by constants A and B. These constants can be found substituting initial conditions into the complete solution. Responses to an unforced system for each ζ regime given initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ are graphed below. Note that the transient response dies out (goes to zero) for all but the undamped system.



3.2 Particular Solution (Steady State Response)

The particular solution of $m\ddot{x} + c\dot{x} + kx = F(t)$ depends on the nature of $F(t)$.

Free Response: If $F(t) = 0$, $x_p(t)$ will also be zero:

$$\boxed{x_p(t) = 0}$$

Step Response: If $F(t) = F_0$ (constant), $x_p(t)$ will also be constant:

$$\boxed{x_p(t) = \frac{F_0}{k}}$$

Harmonic Response: If $F(t) = F_0 \sin(\omega t)$, $x_p(t)$ will oscillate at the same frequency:

$$\boxed{x_p(t) = A\sin(\omega t) + B\cos(\omega t)}$$

Note that this can be written as a single sinusoid using Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and the polar representation of a complex number, $A + iB = \sqrt{A^2+B^2}\,\exp[i\,\text{atan2}\,(B,A)]$. We are using the function atan2(y,x) which gives angles for the entire $[0,2\pi)$ domain, as opposed to $\arctan(y/x)$ which only returns angles in the domain $[-\pi/2,\pi/2]$. See wikipedia for further explanation.

$$\begin{aligned}
A\sin(\omega t) + B\cos(\omega t) &= \text{Im}\{(A+iB)[\cos(\omega t) + i\sin(\omega t)]\} \\
&= \text{Im}\{(A+iB)e^{i\omega t}\} \\
&= \text{Im}\{(\sqrt{A^2+B^2}\,e^{i\phi})e^{i\omega t}\} && \text{for} \qquad \phi = \text{atan2}\,(B,A) \\
&= \text{Im}\{X\,e^{i(\omega t+\phi)}\} && \text{for} \qquad X = \sqrt{A^2+B^2} \\
&= X\,\text{Im}\{\cos(\omega t+\phi) + i\sin(\omega t+\phi)\} \\
&= X\sin(\omega t+\phi)
\end{aligned}$$

We will derive the values of the amplitude X and phase ϕ constants below.

3.3 Complete Solution

To solve for the complete solution $x(t) = x_h(t) + x_p(t)$, we can find the values of the constants A and B in the homogeneous equation by using initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

4 Frequency Response

When a system is driven with a harmonic input (oscillatory of the form $F(t) = F_0\sin(\omega t)$), we have seen that the system's steady state response will be a sinusoid of the same frequency as the input, but could have different amplitude and phase ($x_p(t) = X\sin(\omega t+\phi)$). Let us derive what this amplitude and phase must be assuming this form of the solution.

$$m\ddot{x} + c\dot{x} + kx = F_0\sin(\omega t) \qquad \text{for} \qquad x(t) = X\sin(\omega t+\phi)$$

$$X(-\omega^2 m + k)\sin(\omega t+\phi) + X(\omega c)\cos(\omega t+\phi) = F_0\sin(\omega t)$$

$$\text{Im}\{X(-\omega^2 m + k)e^{i(\omega t+\phi)} + X(i\omega c)e^{i(\omega t+\phi)}\} = \text{Im}\{F_0 e^{i\omega t}\}$$

$$X\,\text{Im}\{e^{i\phi}e^{i(\omega t)}(-\omega^2 m + k + i\omega c)\} = F_0\,\text{Im}\{e^{i\omega t}\}$$

$$X\sqrt{(k-m\omega^2)^2 + (\omega c)^2}\;\text{Im}\left\{e^{i\phi}e^{i(\omega t)}\exp\left[i\,\text{atan2}\left(\omega c, k-m\omega^2\right)\right]\right\} = F_0\,\text{Im}\{e^{i\omega t}\}$$

$$X(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (\omega c)^2}} \qquad \phi(\omega) = -\operatorname{atan2}\left(\omega c, k - m\omega^2\right)$$

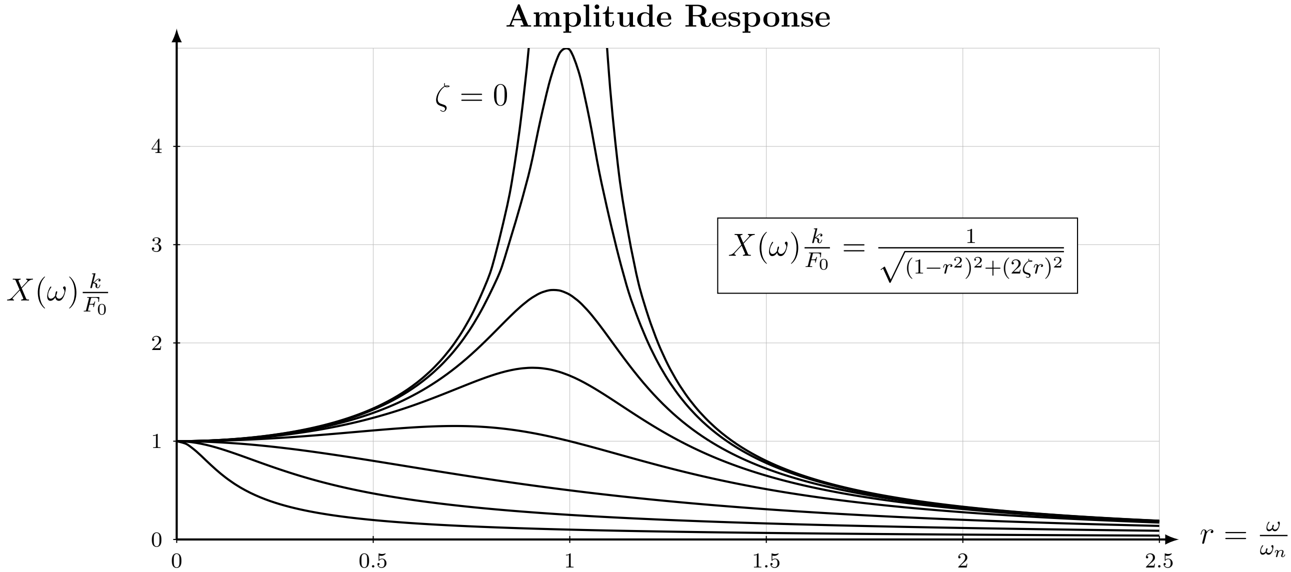
We can also rewrite this amplitude and phase response in terms of ζ and the non-dimensionalized input frequency $r = \omega/\omega_n$:

$$X(r) = \frac{F_0}{k}\frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \phi(r) = -\operatorname{atan2}\left(2\zeta r, 1 - r^2\right)$$

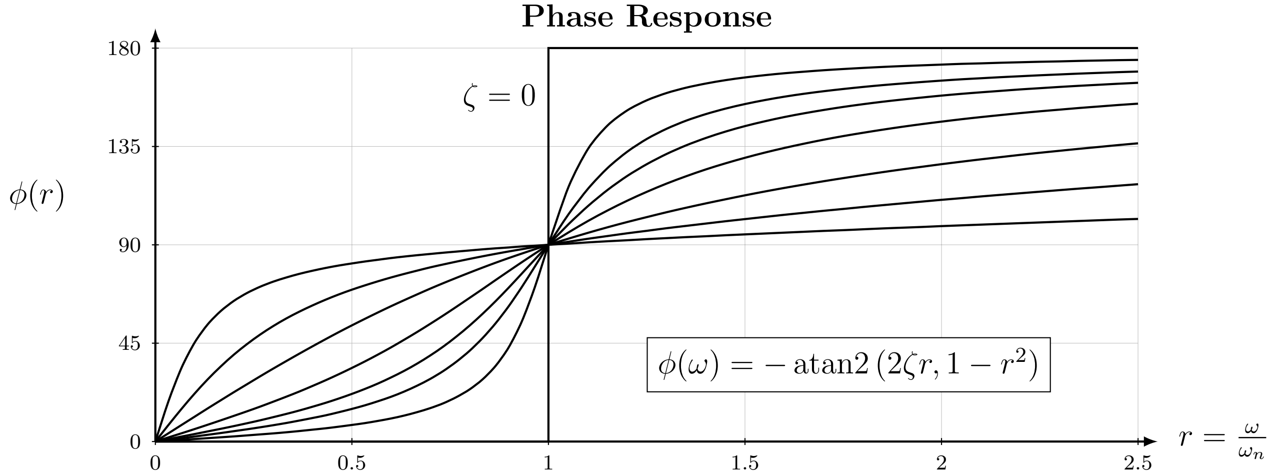
Below are plotted the normalized amplitude response $X(r)\frac{k}{F_0}$ and phase response $\phi(r)$ as functions of the non-dimensional input frequency. Values of ζ starting from zero are (0, 0.1, 0.2, 0.3, 0.5, 1, 2, 5). Note that the peak damped amplitude response occurs to the left of the natural frequency. Indeed the peak is given by:

$$r_{peak} = \sqrt{1 - 2\zeta^2} \qquad \omega_{peak} = \omega_n\sqrt{1 - 2\zeta^2} \qquad X_{peak} = \frac{F_0}{2k\zeta\sqrt{1 - \zeta^2}}$$

Amplitude Response



Phase Response



Frequency response is often useful in non-destructively identifying unknown systems. By forcing the system with a known oscillatory force $F_0 \sin(\omega t)$ for different ω, we can arrive at a dimensional version of the above plots. We can readily find the value of k for the system by seeing that $X(0) = F_0/k$. Also for an underdamped system, we can estimate the natural frequency from the peak frequency to find m. Then we can take any other point on the curve to solve for c.

5 Vibration Modes

In general, when finding the equations of motion for multi-degree of freedom systems, our choice of generalized coordinates will yield a coupled system of equations. Let us examine a general two degree of freedom undamped system with two coupled equations of motion:

$$m_1\ddot{x}_1 + (k_1 + k_3)x_1 - k_3x_2 = 0$$
$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_3x_1 = 0$$

or

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Modes of vibration occur when the entire system is vibrating at the same frequency. Let us then assume a solution of the following form for which a_1, a_2, and ω are unknowns.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(\omega t + \phi)$$

Plugging into our equations of motion yields:

$$\left(-\omega^2 \begin{bmatrix} m_1a_1 \\ m_2a_2 \end{bmatrix} + \begin{bmatrix} (k_1 + k_3) & -k_3 \\ -k_3 & (k_2 + k_3) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) \cancel{\sin(\omega t + \phi)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This equation must hold for all time, thus the term in the parenthesis must be zero. We now have two equations in three unknowns. We can solve for ω by solving each equation for a_1/a_2.

$$\frac{a_1}{a_2} = \frac{k_3}{(k_1 + k_3) - m_1\omega^2} = \frac{(k_2 + k_3) - m_2\omega^2}{k_3}$$

$$k_3^2 = (k_1 + k_3 - m_1\omega^2)(k_2 + k_3 - m_2\omega^2)$$

$$m_1m_2\omega^4 - [m_1k_2 + m_2k_1 + (m_1 + m_2)k_3]\omega + [k_1k_2 + (k_1 + k_2)k_3] = 0$$

$$\omega^2 = \frac{[m_1k_2 + m_2k_1 + (m_1 + m_2)k_3] \pm \sqrt{[m_1k_2 + m_2k_1 + (m_1 + m_2)k_3]^2 - 4m_1m_2[k_1k_2 + (k_1 + k_2)k_3]}}{2m_1m_2}$$

For $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$, $\quad \omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}, \quad \left[\frac{a_1}{a_2}\right]_1 = 1, \quad \left[\frac{a_1}{a_2}\right]_2 = -1$

For $2m_1 = m_2 = 2m$ and $k_1 = k_2 = 0$, $\quad \omega_1 = 0, \quad \omega_2 = \sqrt{\frac{3k}{2m}}, \quad \left[\frac{a_1}{a_2}\right]_1 = 1, \quad \left[\frac{a_1}{a_2}\right]_2 = -2$

The following will not be tested on the final, but is useful in understanding the purpose of studying vibration modes. The amplitude ratios can be thought of as a generalized eigenvectors or normal modes for the system, while the corresponding vibration frequency would be the associated eigenvalue or resonant frequency.

These vibration modes can be thought of as defining a new independent set of generalized coordinates that decouple the equations of motion. It is possible to find new generalized coordinates by summing the original equations of motion in proportions related to the normal modes. For example, for the case where $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$, we can define two new orthogonal generalized coordinates $\eta_1 = x_1 + x_2$ and $\eta_2 = x_1 - x_2$ for which the equations of motion are decoupled:

$$m\ddot{\eta}_1 + k\eta_1 = 0$$

$$m\ddot{\eta}_2 + 3k\eta_2 = 0$$

For the case where $2m_1 = m_2 = 2m$ and $k_1 = k_2 = 0$, we can define two new orthogonal generalized coordinates $\eta_1 = x_1 + 2x_2$ and $\eta_2 = x_1 - x_2$ for which the equations of motion are also decoupled:

$$\ddot{\eta}_1 = 0$$

$$2m\ddot{\eta}_2 + 3k\eta_2 = 0$$

We say that the modes of vibrations are orthogonal to each other because vibration in one mode does not affect the vibration in another. Moreover, all possible motions of a multi-degree of freedom undamped system will be a linear combination of its vibration modes.